Measures and metrics in o-minimal fields I

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O-minimality

Throughout $R = (\mathbb{R}, +, \times, \leqslant, \ldots)$ is an o-minimal expansion of the real field.

Throughout "definable" means "R-definable, possibly with parameters".

Bilipschitz Equivalence

Let (X, d) and (X', d') be metric spaces.

A **bilipschitz equivalence** $(X, d) \to (X', d')$ is a bijection $f : X \to X'$ such that for some $\lambda_1, \lambda_2 > 0$ we have

$$\lambda_1 d(x, y) \leqslant d'(f(x), f(y)) \leqslant \lambda_2 d(x, y) \quad \text{for all } x, y \in X.$$

(X, d) and (X', d) are **bilipschitz equivalent** if there is a bilipschitz equivalence $(X, d) \to (X', d')$.

Definable Metric Spaces

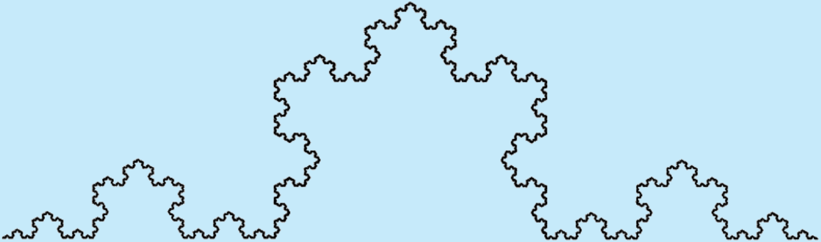
A **definable metric space** is a pair (X, d) where X is a definable set and d is definable metric on X.

A theory of definable metric spaces should be some kind of **tame metric geometry**

Examples:

Any definable set X together with the induced euclidean metric e.

Snowflakes: $([0,1], d)$ with $d(x, x') = |x - x'|^r$ for $r \in (0,1)$.
The Hausdorff dimension of an r-flake is $\frac{1}{r}$.



Carnot Groups

A Carnot Group is a certain kind of nilpotent lie group.
One example is the Heisenberg Group of matrices:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

Carnot groups admit semialgebraic left-invariant metrics.

For the Heisenberg group the metric is of the form:

$$d(A, B) = \|A^{-1}B\|_H$$

where the H-norm of the matrix above is

$$[x^4 + y^4 + z^2]^{\frac{1}{4}}$$

The Hausdorff dimension of the Heisenberg Group is 4.

Topological Dichotomy

Theorem

Let (X, d) be definable. Exactly one of the following holds:

1. *There is an infinite definable $A \subseteq X$ such that (A, d) is discrete.*
2. *There is a definable $Z \subseteq \mathbb{R}^k$ and a definable homeomorphism*

$$(X, d) \to (Z, e).$$

If (X, d) satisfies (i) then the Hausdorff dimension of (X, d) is infinite.

Definable Simplicial Complexes

Let (V, E) be a definable graph.

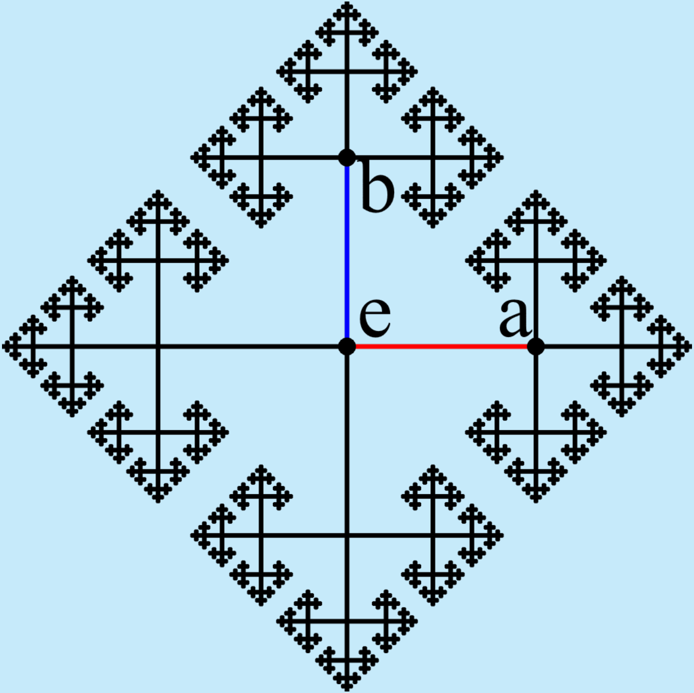
There is a definable metric space which is homeomorphic to the geometric realization of (V, E).

Let V be a definable set and let $f_1, f_2 : V \to V$ be definable functions which generate a free action of a free group on two elements.

We declare $(x, y) \in E$ iff there is a $i \in \{0, 1\}$ such that

$$f_i(x) = y \quad \text{or} \quad f_i(y) = x.$$

Then (V, E) is the disjoint union of continumn many copies of the Cayley graph of a free group on two generators.


b
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Problem

Describe definable metric spaces up to homeomorphism.

Question

Is every definable metric space homeomorphic to a semilinear definable metric space?

"semilinear" means definable in the the reals considered as an ordered vector space over itself.

Metric Dichotomy

Suppose R is polynomially bounded. Let (X, d) be a definable metric space.

Theorem

One of the following holds:

1. *There is a definable $A \subseteq X$ such that (A, d) is definably bilipschitz equivalent to some r-snowflake of the unit interval.*
2. *Almost every $p \in X$ has a neighborhood U such that*

$$\mathrm{id} : (U, d) \to (U, e) \quad \textit{is bilipschitz.}$$

Theorem

Suppose that the Hausdorff dimension of (X, d) is $\dim(X)$. Then almost every $p \in X$ has a neighborhood U such that

$$\mathrm{id} : (U, d) \to (U, e) \quad \textit{is bilipschitz.}$$

Valette's Finiteness Theorem

Suppose that R is polynomially bounded. Let Λ be the field of powers of R.

Theorem (Valette)

There are only $|\Lambda|$-many definable sets up to bilipschitz equivalence. A definable family of sets contains only finitely many elements up to bilipschitz equivalence.

There is a semialgebraic family of metric spaces which contains infinitely many elements up to bilipschitz equivalance.

Theorem (Pansu)

If two Carnot groups are bilipschitz equivalent then they are isomorphic as groups.

Thank you.