Transseries, Hardy fields, and surreal numbers

Lou van den Dries

University of Illinois at Urbana-Champaign

Overview

I. Reminders from Aschenbrenner's talk

II. Remarks on Hardy fields

III. Connection to the surreals

IV. Open problems

(joint work with MATTHIAS ASCHENBRENNER and JORIS VAN DER HOEVEN)

I. Reminders from Aschenbrenner's talk

Main Theorem

We consider $\mathbb{T}$ as a valued ordered differential field, that is, as a structure for the language with the primitives

$$0,\ 1,\ +,\ \cdot,\ \partial \text{ (derivation)},\ \leqslant \text{ (ordering)},\ \preccurlyeq \text{ (dominance)}.$$

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$\mathrm{Th}(\mathbb{T})$ *is axiomatized by the following:*

1. *Liouville closed H-field;*
2. ω*-free;*
3. *newtonian.*

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ω-free: certain pseudo-cauchy sequences have no pseudo-limits. So a model of this theory is never spherically complete. Newtonianity is a kind of differential-henselianity.

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Suppose K is an H-field with $\partial(K) = K$ and K is a directed union of spherically complete grounded H-subfields. Then K is ω-free and newtonian.

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(A kind of analogue to Hensel's Lemma which says that spherically complete valued fields are henselian.)

II. Remarks on Hardy fields

Hardy fields as H-fields

A Hardy field is a field K of germs at $+\infty$ of differentiable functions $f : (a, +\infty) \to \mathbb{R}$ such that the germ of f' also belongs to K. For simplicity, assume also that Hardy fields contain $\mathbb{R}$.

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This is because $\mathrm{Th}(\mathbb{T})$ is the model companion of the theory of H-fields, and has a Hardy field model isomorphic to

$$\mathbb{T}^{\mathrm{da}} := \{f \in \mathbb{T} : f \text{ is d-algebraic}\}.$$

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To answer the question it remains to show that every Hardy field has a newtonian Hardy field extension.

III. Connection to the surreals

No as an *H*-field

Berarducci and Mantova recently equipped Conway's field **No** of surreal numbers with a derivation ∂ that makes it a Liouville closed *H*-field with constant field $\mathbb{R}$.

Moreover, the BM-derivation ∂ respects infinite sums, and is in a certain technical sense the simplest possible derivation on **No** making it an *H*-field with constant field $\mathbb{R}$ and respecting infinite sums.

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Is **No** with the BM-derivation elementarily equivalent to $\mathbb{T}$?

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It is easy to produce spherically complete additive subgroups and subfields of **No**: for any set $S \subseteq$ **No** we have the spherically complete additive subgroup

$$\mathbb{R}[[\omega^S]] := \{a = \sum_{s \in S} r_s \omega^s : \text{ supp } a \text{ is reverse well-ordered}\}$$

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If S has a least element, then $\mathbb{R}[[\omega^S]]$ has a smallest archimedean class. If S is already an additive subgroup, then $\mathbb{R}[[\omega^S]]$ is a spherically complete subfield of **No**.

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So let S be an initial subset of **No**. Then the ordered additive group $\Gamma := \mathbb{R}[[\omega^S]]$ is initial, and so is $K := \mathbb{R}[[\omega^\Gamma]]$. (Ehrlich)

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3. $S = \{0, -1\}$ gives $\Gamma = \mathbb{R} + \mathbb{R}\omega^{-1}$, so $K = \mathbb{R}[[\omega^{\mathbb{R}} \cdot \log(\omega)^{\mathbb{R}}]]$, closed under ∂.

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Let ε be an ε-number, that is, an ordinal such that $\omega^\varepsilon = \varepsilon$. Set

$$S_\varepsilon := \{\text{surreals of length } < \varepsilon\}.$$

Then S_ε is initial, and we can show that the resulting spherically complete subfield K_ε of **No** is closed under ∂. Recall:

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$$S_{\varepsilon} \ := \ \{\text{surreals of length } < \varepsilon\}.$$

Then S_{ε} is initial, and we can show that the resulting spherically complete subfield K_{ε} of **No** is closed under ∂. Recall:

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But the H-field K_{ε} is not grounded, since S_{ε} doesn't have a least element.

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Remedy: take $S^\varepsilon := S_\varepsilon \cup \{-\varepsilon\}$. Then S^ε is still initial, but now has also a least element, namely $-\varepsilon$. Using the fact that K_ε is closed under ∂, it follows that the field K^ε obtained from S^ε is still closed under ∂.

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So we have for each ε-number ε a spherically complete grounded H-subfield K^ε of **No**. Easy to check that **No** is the increasing union of those K^ε.

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Thus **No** with ∂ is elementarily equivalent to $\mathbb{T}$.

Related results

- there is a unique embedding $\mathbb{T} \to$ **No** of exponential fields that is the identity on $\mathbb{R}$ and respects infinite sums; this embedding also respects the derivations and is therefore an elementary embedding of differential fields. (Routine)
- The subfield of **No** consisting of the surreals of countable length is closed under ∂. (Less routine)

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The second result depends on the fact, of independent interest, that for any countable ordinal λ, any well-ordered set of surreals of length $< \lambda$ is countable.

IV. Open Problems

$\mathbb{T}$ as a differential exponential field

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This leads to the obvious question whether $\mathbb{T}$ as a differential exponential field has a reasonable model theory. I am optimistic that this is the case. Recall: exp and ∂ are compatible in the sense that $(\exp f)' = f' \exp f$.

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And what about **No** as a differential exponential field?

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Easy: if A is definably closed set in a model of Th($\mathbb{T}$), then it is an H-subfield of that model.

Does every definable family $(X_f)_{f \in \mathbb{T}^m}$ of (definable) subsets of $\mathbb{T}^n$ have the uniform finiteness property?

That is, given such a family, is there a bound $B \in \mathbb{N}$ such that all finite X_f have size $\leqslant B$?

Uniform finiteness?

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Is there a reasonable dimension theory for definable sets in $\mathbb{T}$?

Set $\ell_0 := x, \ell_1 := \log x, \ldots, \ell_{n+1} = \log \ell_n$. Define

$$\mathbb{T}_{\log} := \bigcup_n \mathbb{R}[[\ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]].$$

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$\mathbb{T}_{\log}$ is a particularly transparent H-subfield of $\mathbb{T}$. It is ω-free and newtonian by the same theorem we used in showing that $\mathbb{T}$ and **No** are ω-free and newtonian.

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But $\mathbb{T}_{\log}$ is **not** Liouville closed. It is power closed: every differential equation $y^\dagger = cf^\dagger$ ($c \in \mathbb{R},\ f \in \mathbb{T}_{\log}$) has a solution, namely $y = f^c$.

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Much of the AHD-work does not use Liouville closednes, but concerns arbitrary ω-free newtonian H-fields, and this gives hope that $\mathbb{T}_{\log}$ also has a reasonable model theory.

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Gehret's Program is to show that the following axiomatizes a complete and model complete theory:

- H-field with real closed constant field;
- ω-free and newtonian;
- closed under powers;
- asymptotic couple $\models$ theory in (1) above;
- axiom from (2) above.

The new axiom in (2) above was suggested by trying to existentially define the **complement** of the existentially definable set $\{f^\dagger :\ f \in \mathbb{T}_{\log}\}$, an $\mathbb{R}$-linear subspace of $\mathbb{T}_{\log}$.

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Gehret noticed that this is possible in the two-sorted structure consisting of $\mathbb{T}_{\log}$ with its asymptotic couple as second sort:

$y \notin \{f^\dagger :\ f \in \mathbb{T}_{\log}\}$ iff there exists a $g \neq 0$ such that $v(y - g^\dagger) \in \Psi^{\downarrow} \setminus \Psi$, where

$$\Psi \ := \ \{v(a^\dagger) :\ a \in \mathbb{T}_{\log}^\times,\ v(a) \neq 0\}$$

is an important definable set in the asymptotic couple of $\mathbb{T}_{\log}$.