APPLICATIONS OF JORDAN FORMS TO SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

SHAUN DEAN

ABSTRACT. The Jordan Canonical Form of a matrix is a very important concept from Linear Algebra. One of its most important applications lies in the solving systems of ordinary differential equations. The Jordan Form results in a useful description of the nilpotent part of a matrix and we will discuss its uses for calculating the matrix exponential. Further we will see how to find a basis of eigenvectors and their use for solving differential equations.
On the other hand finding the Jordan Form of a matrix does not necessarily have to be the best way to solve a differential equation, as finding a basis of eigenvectors, can be very hard.

CONTENTS

1. Fundamental Theorem for Linear Systems

Definition 1.1. Let A be a complex quadratic matrix. For $t \in \mathbb{C}$ we define the matrix exponential as:

$$e^{At} := \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j$$

Lemma 1.2. *If the matrices A and B commute , i.e. $AB = BA$, we get $e^{A+B} = e^A \cdot e^B$*

Proof. Let $AB = BA$.

$$\begin{aligned} e^{A+B} &= \sum_{j=0}^{\infty} \frac{(A+B)^j}{j!} \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{j}{k} \frac{A^k B^{j-k}}{j!} \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{A^k B^{j-k}}{k!(j-k)!} \\ &= \sum_{p=0}^{\infty} \frac{A^p}{p!} \cdot \sum_{q=0}^{\infty} \frac{B^q}{q!} \\ &= e^A \cdot e^B \end{aligned}$$

Here we have used the fact that both e^A and e^B converge absolutely and we can therefore multiply the two series entry by entry. □

Theorem 1.3. Fundamental Theorem for Linear Symstems
Let A be an $n \times n$ matrix with $n \in \mathbb{N}$. For $x_0 \in \mathbb{C}^n$ the initial value problem

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \tag{1.1}$$

has a unique solution given by $x(t) = e^{At}x_0$.

Proof. First we prove that $x(t) = e^{At}x_0$ is in fact a solution of (1.1).

$$\begin{aligned}
x'(t) &= \frac{d}{dt}e^{At}x_0 \\
&= \lim_{h\to 0} \frac{e^{A(t+h)} - e^{At}}{h} \cdot x_0 \\
&= \lim_{h\to 0} \left(\frac{e^{Ah} - I}{h}\right) e^{At} \cdot x_0 \\
&= \lim_{h\to 0} \left(\sum_{j=0}^{\infty} \frac{h^j}{h \cdot j!} A^j - \frac{I}{h}\right) e^{At} \cdot x_0 \\
&= \lim_{h\to 0} \left(\sum_{j=2}^{\infty} \frac{h^{j-1}}{j!} A^j + A + \frac{I}{h} - \frac{I}{h}\right) e^{At} \cdot x_0 \\
&= \left(\lim_{h\to 0} \left(\sum_{j=2}^{\infty} \frac{h^{j-1}}{j!} A^j\right) + A\right) e^{At} \cdot x_0 \\
&= Ae^{At} \cdot x_0 \\
&= A \cdot x(t)
\end{aligned}$$

and $x(0) = I \cdot x_0 = x_0$
Secondly we have to prove the uniqueness of the solution. So let y be any solution to (1.1) and let $z(t) := e^{-At}y(t)$

$$\begin{aligned}
z'(t) &= -Ae^{-At}y(t) + e^{-At}y'(t) \\
&= -Ae^{-At}y(t) + e^{-At}Ay(t) \\
&= 0
\end{aligned}$$

Therefore z is a constant. We also know $z(0) = y(0) = x_0$, so $z(t) = x_0$, so $y(t) = e^{At}x_0$, so $y = x$ and x is the unique solution of (1.1). □

2. Jordan Forms

Definition 2.1. Jordan Form
Let A be a complex matrix with $A \in \mathbb{C}^{n\times n}$ and $n \in \mathbb{N}$, with eigenvalues λ_j, with $j = 1, \dots, d$. Then there exists a basis $\{v_1, \dots, v_n\}$, where v_j, $j = 1, \dots, n$ are eigenvectors of A such that the matrix $P = (v_1, \dots v_n)$ is an invertible matrix with

$$P^{-1}AP = J := \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_d \end{pmatrix} \iff A = PJP^{-1}$$

with

$$J_j := \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ & \lambda_j & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \lambda_j & 1 \\ 0 & & & & \lambda_j \end{pmatrix} \in \mathbb{C}^{m_j \times m_j},\ j = 1, \dots, d,\ \sum_{j=1}^{d} m_j = n$$

Theorem 2.2. Using the Notation from definition 2.1 we can express the solution of (1.1) with: $x(t) = P \begin{pmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_d t} \end{pmatrix} P^{-1}x_0$

and for $j = 1, \dots d$ and eigenvalue λ_j

$$e^{J_j t} = e^{\lambda_j t} \cdot \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m_j-1}}{(m_j-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}$$

Proof. The solution of (1.1) is given by

$$\begin{aligned} x(t) &= e^{At}x_0 \\ &= e^{PJP^{-1}t}x_0 \\ &= \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}(PJP^{-1})^k\right) x_0 \\ &= \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} P \cdot J^k \cdot P^{-1}\right) x_0 \\ &= P \cdot \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} J^k\right) \cdot P^{-1}x_0 \\ &= Pe^{Jt}P^{-1}x_0 \end{aligned}$$

The question is now what does e^{Jt} look like?

$$e^{Jt} = \begin{pmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_d t} \end{pmatrix}$$

For $j \in \{1, \dots, d\}$ let $N_j = \begin{pmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \end{pmatrix} \in \mathbb{C}^{m_j \times m_j}$, so defining $J_j = (\lambda_j \cdot I + N_j)$ and we get

$$e^{J_j t} = e^{(\lambda_j \cdot I + N_j)t} = e^{\lambda_j \cdot I \cdot t} \cdot e^{N_j \cdot t} = e^{\lambda_j \cdot t} \cdot e^{N_j \cdot t}$$

Further we can calculate $e^{N_j \cdot t}$ by looking at the exponents of N_j and we receive:

$$N_j^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & & \\ & & & & 0 \end{pmatrix}, \dots, N_j^{m_j - 1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & \cdots & 0 & 0 \end{pmatrix}, N_j^{m_j} = 0.$$

This means N_j is nilpotent of order m_j , hence

$$e^{N_j t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot N_j^k = \sum_{k=0}^{m_j - 1} \frac{t^k}{k!} \cdot N_j^k = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m_j-1}}{(m_j-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}$$

and therefore

$$e^{J_j t} = e^{\lambda_j t} \cdot e^{N_j t} = e^{\lambda_j t} \cdot \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m_j-1}}{(m_j-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}$$

□

Corollary 2.3. Returning to (1.1) we know that $\dot{x} = Ax \iff x'(t) = PJP^{-1}x(t)$. If we define $x := Py$, $x'(t) = Ax(t)$ is equivalent to
$Py'(t) = P^{-1}JPP^{-1}y(t) \iff y'(t) = Jy(t)$. Therefore

$$\dot{y} = Jy \tag{2.1}$$

We know that a solution for (1.1) is given by $x(t) = Py(t)$, where y is a solution for (2.1)
$\dot{y} = Jy$ can be split up into uncoupled equations $u' = J_j u_j$. Here we know that each coordinate of the solution is now given by a linear combination of functions of the form:
$t^k \cdot e^{\lambda_j t}, \quad (j = 1, \ldots, d,\ k = 0, \ldots, m_j - 1)$

Example 2.4. Regard the initial value problem (1.1) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & 1 \end{pmatrix}$$

$\det(A - \lambda I) = -(\lambda - 1)(\lambda - 2)(\lambda + 2)$.
Therefore we have three eigenvalues, each of multiplicity one:
$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -2$

$$\ker(A - \lambda_1 I) = \ker \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{pmatrix} \implies v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\ker(A - \lambda_2 I) = \ker \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -4 & 4 & -1 \end{pmatrix} \implies v_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\ker(A - \lambda_3 I) = \ker \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{pmatrix} \implies v_3 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Now we can write $A = PJP^{-1}$,

with $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & 4 & 4 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} \frac{4}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & -\frac{1}{4} & \frac{1}{12} \end{pmatrix}$

Further we know the solution is $x(t) = Pe^{Jt}P^{-1}x_0 = P \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} P^{-1}x_0$

$$= \begin{pmatrix} \frac{-3e^{4t}+8e^{3t}+1}{6e^{2t}} & \frac{e^{4t}-1}{4e^{2t}} & \frac{3e^{4t}-4e^{3t}+1}{12e^{2*t}} \\ \frac{-3e^{4t}+4e^{3t}-1}{3e^{2t}} & \frac{e^{4t}+1}{2e^{2t}} & \frac{3e^{4t}-2e^{3t}-1}{6e^{2t}} \\ \frac{-6e^{4t}+4e^{3t}+2}{3e^{2t}} & \frac{e^{4t}-1}{e^{2t}} & \frac{3e^{4t}-e^{3t}+1}{3e^{2t}} \end{pmatrix} x_0$$

If for example $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ then $x(t) = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}$

Definition 2.5. Let λ be an eigenvalue of the $n \times n$ matrix A of multiplicity n. We define the deficiency indices as

$$\delta_k := \dim(\ker(A - \lambda I)^k) \quad (k = 1, \ldots, n)$$

To find δ_k we can use the Gaussian reduction on $(A - \lambda I)^k$ and δ_k is the number of zero rows in the reduced echelon form of $(A - \lambda I)^k$

Let ν_k be the number of Jordan blocks of size $k \times k$ in the Jordan canonical from of A.

$$\sum_{k=1}^{n} \nu_k = \delta_1 \quad \text{is the number of Jordan Blocks in } A$$

With [4] p. 124 we get

$$\begin{aligned}
\delta_2 &= \nu_1 + 2\sum_{k=2}^{n} \nu_k \\
\delta_3 &= \nu_1 + 2\nu_2 + 3\sum_{k=3}^{n} \nu_k \\
&\vdots \\
\delta_{n-1} &= \nu_1 + 2\nu_2 + 3\nu_3 + \ldots + (n-1)\nu_{n-1} + (n-1)\nu_n \\
\delta_n &= \sum_{k=1}^{n} k \cdot \nu_k
\end{aligned}$$

and therefore we get

$$\begin{aligned}
\nu_1 &= 2\delta_1 - \delta_2 \\
\nu_k &= 2\delta_k - \delta_{k+1} - \delta_{k-1} \quad (1 < k < n) \\
\nu_n &= \delta_n - \delta_{n-1}
\end{aligned}$$

In the following we present an algorithm for finding a basis B of eigenvectors such that the $n \times n$ matrix A with the eigenvalue λ of multiplicity n assumes its Jordan Canonical Form J, with respect to the basis B. (cf. [1] p. 43-44)

Step 1) Find a basis $\{v_j^{(1)}\}_{j=1}^{\delta_1}$ for $\ker(A-\lambda I)$. This means we have to find a linearly independent set of eigenvectors corresponding to λ

Step 2) If $\delta_2 > \delta_1$, choose a basis $\{V_j^{(1)}\}_{j=1}^{\delta_1}$ for $\ker(A-\lambda I)$ such that

$$(A - \lambda I)v_j^{(2)} = V_j^{(1)}$$

has $\delta_2 - \delta_1$ linearly independent solutions $v_j^{(2)}$ for $j = 1, \ldots, \delta_2 - \delta_1$. Then

$$\{v_j^{(2)}\}_{j=1}^{\delta_2} = \{V_j^{(1)}\}_{j=1}^{\delta_1} \cup \{v_j^{(2)}\}_{j=1}^{\delta_2-\delta_1}$$

is a basis for $\ker(A - \lambda I)^2$

Step 3) If $\delta_3 > \delta_2$ repeat Step 2) such that $(A - \lambda I)v_j^{(3)} = V_j^{(2)}$ has $\delta_3 - \delta_2$ linearly independent solutions.

Step 4) Continue process until $\delta_k = n$ and obtain a basis $B = \{v_j^{(k)}\}_{j=1}^{n}$

P is then obtained by ordering the eigenvectors with $v_j^{(i)}$ satsfying $(A - \lambda I)v_j^{(i)} = V_j^{(i-1)}$, listed after the entry $= V_j^{(i-1)}$.

Example 2.6. Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \Longrightarrow \det(A - \lambda I) = -(\lambda - 2)^3$$

Therefore we have the eigenvalue $\lambda = 2$ of multiplicity three.

$$\delta_1 = \dim(\ker(A - \lambda I)) = \dim\left(\ker\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\right) = 2$$

Here we can easily find two eigenvectors $v_1^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $v_2^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

As $\delta_1 \neq 3$ we have to proceed with step two of the algorithm. For this we have to look at $\ker(A - \lambda I)^2 = \ker(0)$, so we can choose $v_1^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Further this choice now has to satisfy $(A - \lambda I)v_1^{(2)} = V_1^{(1)}$. As $(A - \lambda I)v_1^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ we have to choose $V_1^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Now we can choose $V_2^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Therefore we receive

$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and for the representation of J we used $\nu_1 = 2\delta_1 - \delta_2 = 1$ and $\nu_2 = 2\delta_2 - \delta_3 - \delta_1 = 1$

For the solution of (1.1) we get $x(t) = P\begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}P^{-1}x_0$

Example 2.7. Let

$$A = \begin{pmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Longrightarrow \det(A - \lambda I) = (\lambda - 1)^4$$

$(A-\lambda I) = \begin{pmatrix} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and using Gaussian reduction we obtain $\delta_1 = 2$ and $v_1^{(1)} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_2^{(1)} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, which span $ker(A - \lambda I)$

Next $\delta_2 = \dim(\ker(A - \lambda I)^2) = \dim\left(\ker\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = 3$ and we get

$v_1^{(2)} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

As $(A - \lambda I)v_1^2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = v_1^{(1)}$ we can leave $V_1^{(1)} = v_1^{(1)}$, so with $V_2^{(1)} = v_2^{(1)}$ we get $\{V_1^{(1)}, v_1^{(2)}, V_2^{(1)}\}$ spans $\ker(I - \lambda I)^2$.

$\delta_3 = \dim(\ker(A - \lambda I)^3) = \dim(\ker(0)) = 4$ and we get $v_1^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

If we now multiply $(A - \lambda I)v_1^{(3)} = v_1^{(2)}$ so we can choose $V_1^{(2)} = v_1^{(2)}$ and therefore $\{V_1^{(1)}, V_1^{(2)}, v_1^{(3)}, V_2^{(1)}\}$ are a basis of $\ker(A - \lambda I)^3$.
Because $\nu_1 = 2\delta_1 - \delta_2 = 1$, $\nu_2 = \delta_2 - \delta_3 - \delta_1 = 0$ and $\nu_3 = 2\delta_3 - \delta_4 - \delta_2 = 1$

$P = \begin{pmatrix} -1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

The solution to the initial value problem (1.1) is then given by

$$x(t) = Pe^t \begin{pmatrix} 1 & t & \frac{t^2}{2} & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^{-1}x_0$$

3. Further applications of Jordan forms

Consider the equation

$$0 = y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_1 y'(t) + a_0 y(t) \tag{3.1}$$

We can transform this higher order scalar linear equation into an equation of order one, using $x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$. Now (3.1) is equivalent to $\dot{x} = Ax$ with

$$A = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

The characteristic polynomial of A is $det(A - \lambda I) = \lambda^n - a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$
Using this it can be shown that the following functions form a basis of the solutions for (3.1):
$y_{j,k}(t) = t^k e^{\lambda_j t}$ for $j = 1 \ldots d, 0 \leq k < m_j$ with m_j the multiplicity of eigenvalue λ_j.
Proof of the above is given in [4] p. 138-139.

References

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