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Skew-symmetric bilinear forms

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Abstract:
This talk should give an outline to the properties of skew-symmetric bilinear forms. We will explore the equivalent representation matrix and investigate its properties. Further we will investigate in the subspaces that can be spanned by the skew-symmetric bilinear forms and the orthogonality of these spaces. At the end we will address the creation of skew-symmetric bilinear forms from linear forms.

1 Definitions:

1.1 Fields, Spaces

Let K be a field, let V, U, W be vector spaces over that field.

1.2 Bilinearity

Let $\alpha, \alpha_1, \alpha_2 \in U$ and $\beta, \beta_1, \beta_2 \in V$. A bilinear map is a function

$$f : U \times V \to W$$

such that

(i) $f(\alpha_1 + c\alpha_2, \beta) = f(\alpha_1, \beta) + cf(\alpha_2, \beta)$

(ii) $f(\alpha, \beta_1 + c\beta_2) = f(\alpha, \beta_1) + cf(\alpha, \beta_2)$

A bilinear form is a bilinear map, with $f : V \times V \to K$.

1.3 Symmetric bilinear form

Let $\alpha, \beta \in V$. A Symmetric bilinear form is a bilinear form, with:

$$f(\alpha, \beta) = f(\beta, \alpha)$$

1.4 Alternating bilinear form

Let $\alpha \in V$. A, alternating bilinear form is a bilinear form with:

$$f(\alpha, \alpha) = 0$$

1.5 Skew-symmetric bilinear form

Let $\alpha, \beta \in V$. A skew-symmetric bilinear form is a bilinear form with:

$$f(\alpha, \beta) = -f(\beta, \alpha)$$

2 Skew-symmetric forms

2.1 Skew-symmetric ⇔ symmetric if and only if $\mathrm{char}(K) = 2$

Let f be a skew-symmetric bilinear form f is **symmetric** if and only if $\mathrm{char}(K) = 2$.
Proof: Let f be any skew-symmetric bilinear form over a vector space V and a field K. This implies

$$f(\alpha, \beta) = -f(\alpha, \beta) \qquad \alpha, \beta \in V$$

it follows either a) or b) because of the characteristic

(a)

$$f(\alpha, \beta) = 0$$

(b)

$$f(\alpha, \beta) = 1$$

The form is in both cases skew-symmetric. In case (a) it is trivial and in case (b) it holds $1 = -1$. The same argument is true for the opposite direction.

2.2 Skew-symmetric ⇔ alternating if and only if $\mathrm{char}(K) \neq 2$

Let f be a skew-symmetric bilinear form f is **alternating** if and only if $\mathrm{char}(K) \neq 2$.
Proof: Let f be any skew-symmetric bilinear form over a vector space V and a field K. This implies

$$f(\alpha, \alpha) = -f(\alpha, \alpha) = 0 \qquad \forall \alpha \in V$$

So f is alternating.

Suppose now f is a alternating. $\alpha, \beta \in V$.

$$\begin{aligned} 0 &= \underbrace{f(\alpha+\beta, \alpha+\beta)}_{0} - \underbrace{f(\beta, \beta)}_{0} - \underbrace{f(\alpha, \alpha)}_{0} \\ &= f(\alpha, \alpha) + f(\alpha, \beta) + f(\beta, \alpha) + f(\beta, \beta) - f(\beta, \beta) - f(\alpha, \alpha) \\ &= f(\alpha, \beta) + f(\beta, \alpha) \end{aligned}$$

It follows $f(\alpha, \beta) = -f(\beta, \alpha)$ for any $\alpha, \beta \in V$, so f is skew-symmetric.

3 Representation Matrix

3.1 Definition representation matrix

Let $\mathcal{B} = \{x_1, \ldots, x_n\}$ be our ordered basis of V, s be any bilinear form.

$$([s]_\mathcal{B})_{i,j} := s(x_i, x_j)$$

3.2 Definition skew-symmetric matrices

The matrix A is skew-symmetric if and only if

$$A^T = -A$$

Remark
The representation matrix of a bilinear form s is skew-symmetric if and only if s is skew-symmetric.

$$([s]_\mathcal{B})_{ij} := s(x_i, x_j) = -s(x_j, x_i) = -([s]_\mathcal{B})_{ji}$$

3.3 Remark on determinants

Let A be a $n \times n$ skew-symmetric matrix. Then:

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$$

So if n is odd the determinant is 0 unless the underlying field has characteristic 2. We will see what happens when n is even later.

4 Subspaces

4.1 Subspace of the skew-symmetric bilinear forms

The set of all skew-symmetric bilinear forms creates a subspace of $L(V, V, K)$ on any vector space V and field K.
Proof: We know that all bilinear forms define a vector space $L_2(V)$
We only need to show closure with scalar multiplication and addition with itself.
Let f, g be skew-symmetric bilinear forms. $\alpha, \beta \in V$

$$(f+g)(\alpha,\beta) = f(\alpha,\beta) + g(\alpha,\beta) = -f(\beta,\alpha) - g(\beta,\alpha) = -(f+g)(\beta,\alpha)$$

Decomposition of this space
$L(V, V, K)$ is the direct sum of the subspace of skew-symmetric bilinear forms ($ssL(V, V, K)$) and the symmetric bilinear forms ($sL(V, V, K)$).
Proof: Show $ssL(V, V, K) \cap sL(V, V, K) = \{0\}$. Let $f \in ssL(V, V, K) \cap sL(V, V, K)$. It holds for $\alpha, \beta \in V$:

$$f(\alpha,\beta) = f(\beta,\alpha) = -f(\alpha,\beta) = -f(\beta,\alpha)$$

This is always true if $\mathrm{char}(K) = 2$. In any other case it implies $f = 0$. Now choose a $f \in L(V, V, K)$ and show that there exists $g \in sL(V, V, K)$ and $h \in ssL(V, V, K)$. For $\alpha, \beta \in V$ let:

$$g(\alpha,\beta) = \frac{1}{2}\left(f(\alpha,\beta) + f(\beta,\alpha)\right)$$
$$h(\alpha,\beta) = \frac{1}{2}\left(f(\alpha,\beta) - f(\beta,\alpha)\right)$$

it follows:

$$g(\alpha,\beta) = \frac{1}{2}\left(f(\alpha,\beta) + f(\beta,\alpha)\right) = \frac{1}{2}\left(f(\beta,\alpha) + f(\alpha,\beta)\right) = g(\beta,\alpha)$$
$$h(\alpha,\beta) = \frac{1}{2}\left(f(\alpha,\beta) - f(\beta,\alpha)\right) = -\frac{1}{2}\left(f(\beta,\alpha) - f(\alpha,\beta)\right) = -h(\beta,\alpha)$$

and

$$f(\alpha,\beta) = \frac{1}{2}\left(f(\alpha,\beta) + f(\beta,\alpha)\right) + \frac{1}{2}\left(f(\alpha,\beta) - f(\beta,\alpha)\right) = g(\alpha,\beta) + h(\alpha,\beta)$$

So

$$L(V, V, K) = sL(V, V, K) \oplus ssL(V, V, K)$$

4.2 Orthogonal decomposition lemma

Let f be a fixed non zero skew-symmetric bilinear form, let $\alpha, \beta \in V$ be vectors such $f(\alpha, \beta) = 1$ (It is clear that α and β are linearly independent). Let $\gamma \in \mathrm{span}(\alpha, \beta)$, with $\gamma = c\alpha + d\beta$, $c, d \in K$.

$$f(\gamma,\alpha) = f(c\alpha + d\beta, \alpha) = cf(\alpha,\alpha) + df(\beta,\alpha) = -d$$
$$f(\gamma,\beta) = f(c\alpha + d\beta, \beta) = cf(\alpha,\beta) + df(\beta,\beta) = c$$

so

$$\gamma = f(\gamma,\beta)\alpha - f(\gamma,\alpha)\beta$$

Because of the linear independence of α and β if $\gamma = 0$ it follows that $c = d = 0$ so $f(\gamma, \beta) = f(\gamma, \alpha) = 0$.
Let $W = \mathrm{span}(\alpha, \beta)$ and let $W^\perp = \{x \in V | \forall \gamma \in W : f(x, \gamma) = 0\}$ we claim that $V = W \oplus W^\perp$.

Choose $y \in W \cap W^\perp$ for all $\gamma \in W$ applies $f(y, \gamma) = 0$ but there is also $e, f \in K$ with $y = e\alpha + g\beta$, if we use γ like above it follows:

$$f(y, \gamma) = f(e\alpha + g\beta, c\alpha + d\beta) = ecf(\alpha, \alpha) + edf(\alpha, \beta) + gcf(\beta, \alpha) + gdf(\beta, \beta) = ec - gd = 0$$

So $ec = gd$ for all $c, d \in K$. So $e = g = 0$ and therefore $y = 0$ so $\{0\} = W \cap W^\perp$

Now show that any vector $x \in V$ has the form $x = \delta + \gamma$, with $\delta \in W^\perp$ and $\gamma \in W$.
Let $\gamma = f(x, \beta)\alpha - f(x, \alpha)\beta$ and $\delta = \varepsilon - x$.
$\gamma \in W$ is clear. For δ holds:

$$\begin{aligned} f(\delta, \alpha) &= f(x - f(x, \beta)\alpha + f(x, \alpha)\beta, \alpha) \\ &= f(x, \alpha) + f(x, \alpha)f(\beta, \alpha) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(\delta, \beta) &= f(x - f(x, \beta)\alpha + f(x, \alpha)\beta, \beta) \\ &= f(x, \beta) - f(x, \beta)f(\alpha, \beta) \\ &= 0 \end{aligned}$$

Thus $\delta \in W^\perp$. Therefore $V = W \oplus W^\perp$.

4.3 Theorem on further decomposition

The finite dimensional vector space V has a finite number of orthogonal subspaces $W_0, \ldots, W_k$ $k \in \mathbb{N}$ with $V = W_1 \oplus \cdots \oplus W_k \oplus W_0$.
Proof:
We will use W, $W^\perp$ and f from above and investigate the restriction of f to $W^\perp$.
This restriction could be the zero form. If not we can find vectors $\alpha_1, \beta_1 \in W^\perp$ with $f(\alpha', \beta') = 1$. Let these two vectors span another subspace W' and let the be complement $W_0 = \left\{\delta \in W^\perp \mid f(\alpha', \delta) = f(\beta', \delta) = 0\right\}$.

$$V = W \oplus W' \oplus W_0$$

If f is still not the zero form on W_0 we can divide W_0 further. If we repeat we get an finite number of pairs:

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_k, \beta_k)$$

with their related spans $W_1, \ldots, W_k$ for $k \in \mathbb{N}$.

$$V = W_1 \oplus \cdots \oplus W_k \oplus W_0$$

Remark
The decomposition depends on f so k differs to other forms. Therefore the decomposition of V is not unique. But the decomposition may also change if we choose other α_1, β_1.

4.4 Representation matrix of skew-symmetric bilinear forms

There exists a basis of V such that the representation matrix of f relative to this base has the form:

$$\operatorname{diag}\{S, S, \ldots, S, 0, \ldots, 0\}$$

were

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Proof: If f is again the zero form the result is trivial. In any other case we use the vectors $\alpha_1, \beta_1, \cdots \alpha_k, \beta_k$ that span the orthogonal subspaces we determent above. These vectors are all linearly independent because:

1. if α_i, β_i were dependent $f(\alpha_i, \beta_i) = 0$
2. if α_i were dependent to the other vectors the sum of the space in which α_i is in wouldn't be direct.

We now add the vectors $\gamma_1, \ldots, \gamma_s$ which are a basis of W_0 so

$$(\alpha_1, \beta_1, \cdots, \alpha_k, \beta_k, \gamma_1, \cdots, \gamma_s)$$

is a basis for V. Now the representation matrix of f to this basis looks like this

$$\begin{pmatrix}
f(\alpha_1,\alpha_1) & f(\alpha_1,\beta_1) & f(\alpha_1,\alpha_2) & f(\alpha_1,\beta_2) & \cdots & 0 & 0 & 0 & \cdots & 0 \\
f(\beta_1,\alpha_1) & f(\beta_1,\beta_1) & f(\beta_1,\alpha_2) & f(\beta_1,\beta_2) & \cdots & 0 & 0 & 0 & \cdots & 0 \\
f(\alpha_2,\alpha_1) & f(\alpha_2,\beta_1) & f(\alpha_2,\alpha_2) & f(\alpha_2,\beta_2) & \cdots & 0 & 0 & 0 & \cdots & 0 \\
f(\beta_2,\alpha_1) & f(\beta_2,\beta_1) & f(\beta_2,\alpha_2) & f(\beta_2,\beta_2) & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & f(\alpha_k,\alpha_k) & f(\alpha_k,\beta_k) & f(\alpha_1,\gamma_1) & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & f(\beta_k,\alpha_k) & f(\beta_k,\beta_k) & f(\beta_k,\gamma_1) & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & f(\gamma_1,\alpha_k) & f(\gamma_1,\beta_k) & f(\gamma_1,\gamma_1) & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & f(\gamma_s,\gamma_s)
\end{pmatrix}$$

Which is the announced form. The rank of this matrix is $2k$
k is dependent on the exact decomposition from above and the decomposition is not unique.

Remark on the determinant of this matrix
To calculate the determinant we change every second vector so we get a diagonal matrix with if $2k = n$ we get $\det(s) = 1$ otherwise there are zeros on the diagonal.

For every skew-symmetric bilinear b form over every basis of V there is a basis change matrix p such that $b = psp^T$ with s in the form from above.

1. if $\dim(V)$ is odd $\det(s) = 0$ so $\det(b) = 0$
2. if $\dim(V)$ is even and $\det(s) = 1$ it follows $\det(b) = \det(p)\det(s)\det(p^T) = \det(p)^2$

5 Linear-forms

5.1 Theorem

Let f be any bilinear form on any vector space and and any field. Let L_f and R_f be the independent mappings of f. f is skew-symmetric if and only if $L_f = -R_f$.
Proof: First let f be skew-symmetric it holds:

$$L_f(u)(v) = f(u, v) = -f(v, u) = -R_f(u)(v)$$

Let $L_f = -R_f$:

$$f(u, v) = L_f(u)(v) = -R_f(u)(v) = -f(v, u)$$

5.2 Theorem

Let L_1, L_2 be linear forms. Let $f(\alpha, \beta) = L_1(\alpha)L_2(\beta) - L_1(\beta)L_2(\alpha)$. f is a skew-symmetric bilinear form. *Proof:*

$$f(\alpha, \beta) = L_1(\alpha)L_2(\beta) - L_1(\beta)L_2(\alpha) = -L_1(\beta)L_2(\alpha) + L_1(\alpha)L_2(\beta) = -f(\beta, \alpha)$$

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