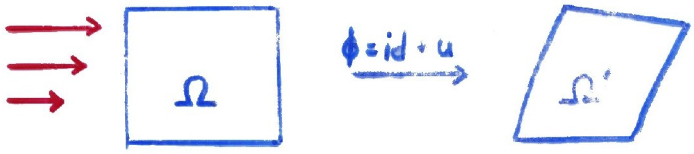
Systeme elliptischer PDGL : Elastizitätsgl. im $\mathbb{R}^2$

1. Problem:



Gesucht : Verschiebung : $\vec{u}$

Verzerrungstensor : $\varepsilon = \varepsilon(\vec{u})$

Spannungstensor : $\sigma = \sigma(\vec{u})$

2. PDGL - System

$$\mu \, \Delta \vec{u} + (\mu + \lambda) \, \nabla (\nabla \cdot \vec{u}) = \vec{f}$$

konstant — Volumenkraft

$$\Longleftrightarrow \quad \mu \begin{pmatrix} \partial_{xx} u_1 + \partial_{yy} u_1 \\ \partial_{xx} u_2 + \partial_{yy} u_2 \end{pmatrix} + (\mu + \lambda) \begin{pmatrix} \partial_{xx} u_1 + \underline{\partial_{xy} u_2} \\ \underline{\partial_{yx} u_1} + \partial_{yy} u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Randbedingungen : $u = 0$ auf Γ_0

$\sigma(u) \cdot \vec{n} = \vec{g}$ auf Γ_1

3. Schwache Formulierung.

$$\int_\Omega [\mu \Delta \vec{u} + (\mu + \lambda) \nabla (\nabla \cdot \vec{u})] \cdot \vec{v} \, d\Omega = \int_\Omega \vec{f} \cdot \vec{v} \, d\Omega$$

$$\Longrightarrow \underbrace{\mu \sum_{i,j=1}^{2} \int_\Omega \partial_i u_j \, \partial_i v_j \, d\Omega + \mu \sum_{i,j=1}^{2} \int_\Omega \partial_i u_j \cdot \partial_j v_i \, d\Omega + \lambda \sum_{i,j=1}^{2} \int_\Omega \partial_i u_i \cdot \partial_j v_j \, d\Omega}_{= a(\vec{u}, \vec{v})}$$

$$= \underbrace{\int_\Omega f_1 v_1 \, d\Omega + \int_\Omega f_2 v_2 \, d\Omega}_{= F(\vec{v})}$$

4. Lösen des PDGL - Systems

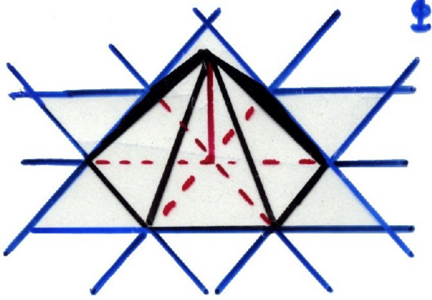
IDEE : Verwende für jede Komponente von $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ die skalaren Basisfunktionen

$\Rightarrow$ Für n Knoten / Freiheitsgrade gibt es $2 \cdot n$ Basisfunktionen

z.B. skalar : $\varphi_1(x,y)$, $\varphi_2(x,y)$

vektoriell : $\Phi_1 := \begin{pmatrix} \varphi_1(x,y) \\ 0 \end{pmatrix}$; $\Phi_2 := \begin{pmatrix} 0 \\ \varphi_1(x,y) \end{pmatrix}$

$\Phi_3 := \begin{pmatrix} \varphi_2(x,y) \\ 0 \end{pmatrix}$; $\Phi_4 := \begin{pmatrix} 0 \\ \varphi_2(x,y) \end{pmatrix}$



5. Diskreter Problem

$$u_h = \sum_{i=1}^{2n} u_i \Phi_i = \begin{pmatrix} \sum_{i=1}^{n} u_{2i-1} & \varphi_i \\ \sum_{i=1}^{n} u_{2i} & \varphi_i \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\Longrightarrow \quad a(u_h, \Phi_i) = F(\Phi_i) \qquad i = 1, \dots, 2n$$

Gesucht : $u_i \in \mathbb{R}^{2n}$

5. Aufbau der Steifigkeitsmatrix

i-te Zeile :

$$a(u_h, \Phi_i) = \sum_{j=1}^{2n} u_j \Big[\mu \int_\Omega \partial_1(\Phi_j)_1 \partial_1(\Phi_i)_1 + \partial_2(\Phi_j)_1 \partial_2(\Phi_i)_1$$

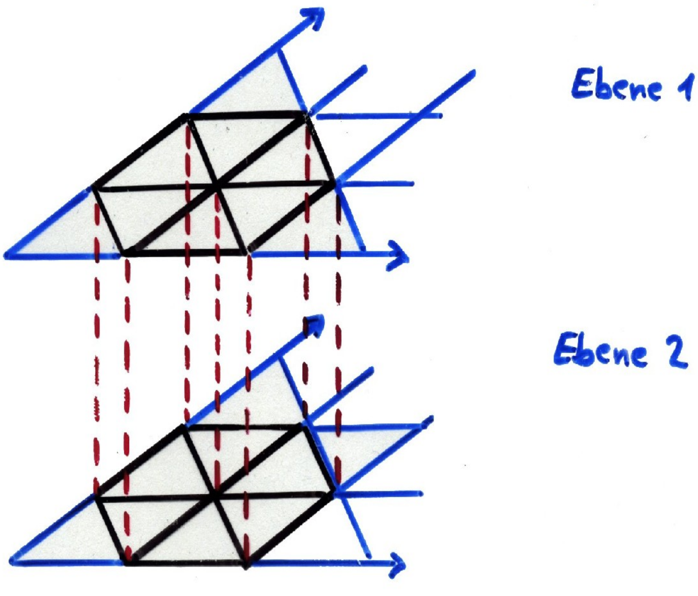
$$+ \partial_1(\Phi_j)_2 \partial_1(\Phi_i)_2 + \partial_2(\Phi_j)_2 \partial_2(\Phi_i) \, d\Omega$$

$$+ \mu \int_\Omega \partial_1(\Phi_j)_1 \partial_1(\Phi_i)_1 + \partial_1(\Phi_j)_2 \partial_2(\Phi_i)_1$$

$$+ \partial_2(\Phi_j)_1 \partial_1(\Phi_i)_2 + \partial_2(\Phi_j)_2 \partial_2(\Phi_i)_2 \, d\Omega$$

$$+ \lambda \int_\Omega \partial_1(\Phi_j)_1 \partial_1(\Phi_i)_1 + \partial_2(\Phi_j)_2 \partial_1(\Phi_i)_1$$

$$+ \partial_1(\Phi_j)_1 \partial_2(\Phi_i)_2 + \partial_2(\Phi_j)_2 \partial_2(\Phi_i)_2 \, d\Omega \Big]$$



6. Realisierung in Matlab

Matlab-Notation für elliptische PDGL:

$$-\nabla\cdot(\underset{\substack{\uparrow\\ c(u)}}{c}\,\nabla u) + \underset{\substack{\uparrow\\ a(u)}}{a}\,u = f$$

$$\Longleftrightarrow \quad \begin{aligned} -\nabla\cdot(c_{11}\nabla u_1) - \nabla\cdot(c_{12}\nabla u_2) + a_{11}u_1 + a_{12}u_2 &= f_1 \\ -\nabla\cdot(c_{21}\nabla u_1) - \nabla\cdot(c_{22}\nabla u_2) + a_{21}u_1 + a_{22}u_2 &= f_2 \end{aligned}$$

unser Problem in dieser Form:

$$-\nabla\cdot\left(\begin{pmatrix} -(2\mu+\lambda) & 0 \\ 0 & -\mu \end{pmatrix}\cdot\nabla u_1\right) - \nabla\cdot\left(\begin{pmatrix} 0 & -\frac{1}{2}(\mu+\lambda) \\ -\frac{1}{2}(\mu+\lambda) & 0 \end{pmatrix}\cdot\nabla u_2\right) = f_1$$

$$-\nabla\left(\begin{pmatrix} 0 & -\frac{1}{2}(\mu+\lambda) \\ -\frac{1}{2}(\mu+\lambda) & 0 \end{pmatrix}\cdot\nabla u_1\right) - \nabla\cdot\left(\begin{pmatrix} -\mu & 0 \\ 0 & -(2\mu+\lambda) \end{pmatrix}\cdot\nabla u_2\right) = f_2$$