Advanced Topics in Nonlinear Optimization

- Suggested Topics -

Each suggested topic includes literature about the basic mathematical theory as well as an application which allows for a simple numerical implementation. In these applications, we propose very simple model instances in order to allow a focus on the new techniques which are introduced. If you require more information or have a topic of your own in mind, please do not hesitate to contact Dennis Beermann (dennis.beermann@uni-konstanz.de) or Jianjie Lu (jianjie.lu@uni-konstanz.de).

- **Optimal Control of a Partial Differential Equation**
  Optimal control theory considers a Partial Differential Equation containing a so-called control variable u. The solution y of this equation will then of course depend on u, i.e. $y = y(u)$. Optimal control stragies then deal with the task of choosing an optimal control $\bar{u}$ such that the solution $y(\bar{u})$ approximates in some way a desired solution y_d.
  An introduction to the theory is presented in [1], a possible numerical application to consider could be a simple parabolic PDE such as the heat equation.

- **Model Predictive Control of an Ordinary Differential Equation**:
  Model predictive control is a special optimization method to deal with certain time-dependent optimal control problem. Again, a control variable u influences a system (discrete system, ODE, PDE, ...) whose solution $y(u)$ should then behave in a certain, desired way. The difference to 'standard' optimal control strategies is the existence of a finite time horizon T_{hor}: At every time t_0, the optimal solution is predicted only for an upcoming interval $(t_0, t_0 + T_{hor})$. Therefore, the algorithm is more interactive than standard Optimal control strategies by being able to adapt to perturbations and changes in the system. An introduction to the theory can be found in [7].
  An example in application could be given by programming an intelligent thermostat: Given a certain desired temperature curve, how should a room be heated with respect to the current temperature?

- **Mixed-Integer Nonlinear Optimization**
  Mixed-Integer Programming considers constrained optimization problems over compact subsets of $\mathbb{R}^N$, where some of the components of the variables $x \in \mathbb{R}^N$ may only take integer values. From the numerous algorithms available, the very basic Branch-and-Bound method for convex problems would be a possible topic. An introduction can be found in [2].

For the numerical application, one could consider a basic Portfolio optimization problem.

- **Multiobjective Optimization**
  Multiobjective optimization considers optimization problems where the cost function J maps to $\mathbb{R}^m$ with $m > 1$. This way, one can treat optimization problems with regard to multiple objectives, for example in order to minimize costs and maximize economic sustainability at the sime time. An introduction to the theory can be found in [10]

- **Global optimization - Trust region methods**
  Global optimization focusses on finding a global optimum to a constrained optimization problem. For nonconvex problems, this is a nontrivial task since the most common strategies only converge to local extreme points in general. A common technique to achieve this is the Trust Region method, an introduction to which can be found in [3].

- **Optimization problems on networks**
  A common application field for optimization problems are supply problems on grids. The specific challenges which arise here are on the one hand the structure of the graph, which has to be efficiently and transparently implemented to achieve an optimization problem in standard form. On the other hand, there is often much data depending on the grid at hand. Therefore, fast and efficient solvers have to be used for these problems. For an introduction to graph theory, we refer to the first chapters in [8].
  A possible application arises from controlling the pressure in the stationary flow within a gas distribution network. An introduction to the modelling of gas flow can be found in [9].

- **Proper Orthogonal Decomposition for Partial Differential Equations**
  When considering PDE-constrained optimization or optimal control with a space variable $x \in \mathbb{R}^d$, the arising optimization problem is often of very high dimension, especially in cases where $d > 1$. At the same time, the solution space often has a much lower dimension. Model Order Reduction techniques consider a specific solution to the equation and aim at approximating said space. A particular method is called Proper Orthogonal Decomposition, which aims at compressing the information of the solution trajectory into as few basis vectors as possible. An introduction to the topic can be found in [4].
  A simple application may consider the nonlinear heat equation in $2D$ and comparing a discretized solution with a fine mesh to the reduced system.

- **Semismooth Newton method**
  When searching for critical points in optimization problems, the Newton method applied to the gradient is a well-established tool. However, this usually only works for twice differentiable functions. The semismooth Newton method offers an extention to functions which are less smooth, utilizing a more general differentiation concept. An introduction can be found in [5].

- **Variational Inequalities**
  Variational Equalities are widely known, with maybe the most common occurence

in the variational (weak) formulations of Partial Differential Equations. Variational Inequalities, on the other hand, consider similar problems where the equality is replaced by an inequality. The general problem can be formulated as finding an element u in a convex set K such that $\langle F(u), y - x \rangle \geq 0$ holds for all $x \in K$.
A very common application to consider is the obstacle problem, which describes for example the shape u a membrane takes under a force F when it cannot extend beyond a certain given boundary. An introduction to variational inequalities is found in [6].

Literatur

[1] Fredi Tröltzsch: *Optimale Steuerung partieller Differentialgleichungen: Theorie, Verfahren und Anwendungen.* Vieweg+Teubner, 2009

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[3] Jorge Noceal, Stephen J. Wright: *Numerical Optimization.* Springer, 2006

[4] Stefan Volkwein: *Proper Orthogonal Decomposition: Applications in Optimization and Control.* Online document, can be found here

[5] Michael Hintermüller: *Semismooth Newton Methods and Applications.* Online document, can be found here

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[7] Lars Grüne, Jürgen Pannek: *Nonlinear Model Predictive Control: Theory and Algorithms.* Springer, 2011

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[10] Kaisa Miettinen: *Nonlinear multiobjective optimization.* Kluwer Academic Publ., 1998