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Exercises for
Theory and Numerics of Partial Differential Equations

http://www.math.uni-konstanz.de/numerik/personen/beermann/en/teaching

Sheet 12

Deadline: Thursday, 09/02, 3:30pm

Exercise 12.1 (Theory)[1] (10 points)

Consider the following *Semi-linear Parabolic Problem*:

$$\begin{cases} y_t - \Delta y = f(x,t,y) \text{ for all } (x,t) \in Q = \Omega \times (0,T) \\ \frac{\partial y}{\partial n} = 0 \text{ for all } x \in \partial\Omega \ , \ t \in (0,T) \\ y(x,0) = y_0(x) \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$, and $f : Q \times \mathbb{R} \to \mathbb{R}$ satisfies the *Carathéodory Condition*[2]. The so-called Nemytskii Operator is a function $N_f : L^\infty(Q) \to L^\infty(Q)$ defined as follows:

$$[N_f(y)](x,t) = f(x,t,y(x,t)) \tag{2}$$

Suppose, moreover, that $f(x,t,y)$ satisfies the following properties:

i) There exists a constant $K > 0$ such that $|f(x,t,0)| \le K$ for almost $(x,t) \in Q$,

ii) $f(x,t,y)$ is locally Lipschitz continuous in y, i.e. for all constants $M > 0$ there exists a constant $L(M) > 0$ such that for almost $(x,t) \in Q$ and for all $y, z \in [-M, M]$ the following inequality holds:

$$|f(x,t,y) - f(x,t,z)| \le L(M)|y - z| \tag{3}$$

Prove that:

1. The Nemytskii Operator is locally Lipschitz continuous in $L^\infty(Q)$:

$$||N_f(y) - N_f(z)||_{L^\infty(Q)} \le L(M)||y - z||_{L^\infty(Q)}$$

for all $y, z \in L^\infty(Q)$, with $||y||_{L^\infty} \le M$ and $||z||_{L^\infty} \le M$,

2. If $f(x,t,y)$ is differentiable in y and f_y satisfies i) and ii), then the Nemytskii Operator N_f is Fréchet-differentiable in $L^\infty(Q)$ and for all $h \in L^\infty(Q)$ satisfies $[N_f'(y)h](x,t) = f_y(x,t,y(x,t))h(x,t)$ for almost every $(x,t) \in Q$.[3]

[1] Notice that we put the theory before the programming part only for this sheet, because it is important that you make first this exercise to understand the Matlab one.

[2] A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the *Carathéodory Condition* if $f(x,t,y)$ is a continuous function of y for almost all $(x,t) \in Q$ and a measurable function of (x,t) for all $y \in \mathbb{R}$.

[3] With f_y we indicate the derivative of f respect to y: $\frac{\partial f}{\partial y}$.

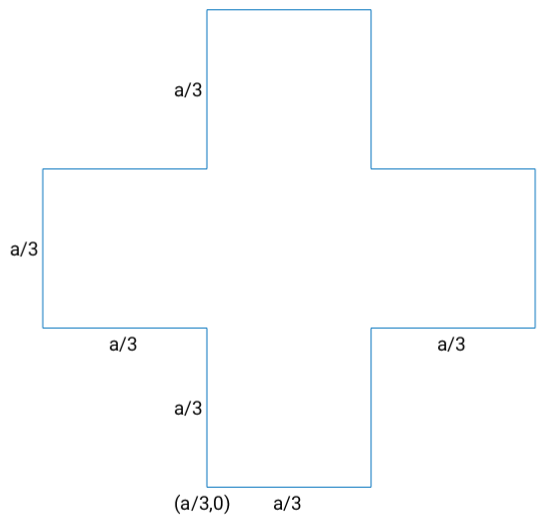
Exercise 12.2 (Matlab) (10 points)

Please follow the *programming guidelines* that can be download under the above url.

Using MATLAB *PDE Toolbox*, solve the following Semiparabolic Problem:

$$\begin{cases} y_t - \Delta y = f(x,t,y) \text{ for all } (x,t) \in Q = \Omega \times (0,T) \\ \frac{\partial y}{\partial n} = 0 \text{ for all } x \in \partial\Omega \text{ , } t \in (0,T) \\ y(x,0) = y_0(x) \end{cases} \tag{4}$$

where $f : Q \times \mathbb{R} \to \mathbb{R}$ satisfies the properties of Exercise 12.1 and $\Omega \subset \mathbb{R}^2$ is given by the interior of the blue line, that depends on the parameter $a > 0$ in $\mathbb{R}$ as shown in figure:



As in the previous Sheet, declare a as global parameter in your main script and make it available in each function. In order to solve the problem follow these steps:

1. **Geometry Implementation:** Write a function `geometryFunction.m` to describe the geometry of Ω by using a suitable analytical boundary representation. Especially focus on the various way that this function will be called by the PDE toolbox (0,1,2 inputs, bs scalar or a vector,...) Then use the command `pdegplot('geometryFunction')` to test your results.

The following two points should be solved in a script.
Do not use point-and-click for these!

2. **PDE specification:** Specify the PDE coefficients in (4) and generate a mesh with maximum element size 0.05. Visualize the mesh.

3. **PDE solving:** Solve the problem for the following choices of $f(x,t,y)$ and $y_0(x)$:

   - $f(x,t,y) = -y^3$, $y_0(x) = 10^{-1}(x_1 + x_2)$ with $x = (x_1, x_2) \in \Omega$;
   - $f(x,t,y) = -e^y - t$, $y_0(x) = 2\pi(\cos(2\pi x_1) + \sin(2\pi x_2))$;
   - $f(x,t,y) = -cos(y) - 2\pi \sin(2\pi x_1)\cos(2\pi x_2)$, $y_0(x) = a$;

   Write a function `solve_semiparabolic_problem` where the problem is solved with the following step:

   (a) Specify the boundary condition and the PDE coefficients,

   (b) Generate the FE matrices with the command `assembleFEMatrices` for the linear part,

   (c) Derive the weak formulation of the non-linear part,

   (d) In each time step, solve the resulting problem with *Newton's Method*, where for time discretization we choose *Implicit Euler Scheme*,

(e) Plot the the time evolution of the solution $y(x,t)$ with $t \in [0,1]$ and $x \in \Omega$.

In order to clarify the exercise, we clarify the steps (c) and (d):

(c) For computing the weak formulation of the non-linear part, we need to go back to the theory. Let be V_h the FE space with $\dim(V_h) = l$ and $\{\phi_1, ..., \phi_l\}$ a basis of V_h, so we can decompose $f(x,t,y)$ in V_h as:

$$f(x,t,y) = \sum_{i=1}^{l} f_i(t,y)\phi_i(x)$$

where $f_i(t,y)$ is the corresponding coefficient of the base ϕ_i.[4] In order to have the coefficient Nlw_j of the weak formulation of non-linear part we have to compute:

$$Nlw_j = \int_\Omega f(x,t,y)\phi_j(x)dx = \sum_{i=1}^{l} f_i(t,y) \int_\Omega \phi_i(x)\phi_j(x)$$

so we have, for the numeric part, that `Nlw= M*Nl` where `M` is the mass matrix and `Nl` is the vector of the coefficient $f_i(t,y)$.

(d) Because of the non-linear part, we need to use the *Newton Method* for computing the solution of the FE system. Also think about what a good initial guess for the method could be.

[4] Numerically speaking, $f_i(t,y)$ is the values of $f(x,t,y)$ computed on the i-th node of the FE mesh