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# Exercises for
# Theory and Numerics of Partial Differential Equations

http://www.math.uni-konstanz.de/numerik/personen/beermann/en/teaching

## Sheet 9

**Deadline: Thursday, 19/01, 5pm**

**Exercise 9.1** (Matlab) (10 points)

**Assorted files:** `plot_solution.m`
Please follow the *programming guidelines* that can be downloaded under the above url.

In this exercise, you are supposed to write two functions that build and solve the linear system generated by the 5-Point Finite Difference Scheme for the Laplace equation:

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

The exercise is divided in two parts:

1. In the *building part*, the matrix and the right-hand side of the linear system have to be built. This is supposed to be done by calling a function

   `[A,b] = build_linear_system(grid,f,g)`

   which takes three arguments:

   (a) `grid` is the structure generated using the function `compute_fd_grid` from Exercise 8.1

   (b) `f` and `g` belong to the class `function_handle` and return, respectively, the value of $\mathtt{f}_{ij}$ = `f(x1(i),x2(j))` and $\mathtt{g}_{ij}$ = `g(x1(i),x2(j))`, where `(x1(i),x2(j))` is a node of the grid.

   and gives as outputs `A` and `b`, which are, respectively, the matrix and the right-hand side of the linear system.

2. In the *solving part*, the system $Au = b$ has to be solved. This is supposed to be done by calling a function

   `u = solve_linear_system(grid, A, b, q1, q2, q3)`

   where `q1`,`q2`,`q3` $\in \{0; 1\}$ represent the answers to the following questions to an imaginary user:

   (a) Do you want to convert the matrix `A` in a sparse form? Input: 1=yes 0=no.

   (b) Do you want to use LU factorization for solving the system? Input: 1=yes 0=no.

   (c) Do you want to use the Cholesky[1] method for solving the system? Input: 1=yes 0=no.

   The function is supposed to be able to handle the user's requests. If the user does not want to solve the system in a particular way, i.e. `q2` and `q3` are both 0, then it is solved with the simple `A\b` command. The function `solve_linear_system` also has to display the overall computational time for getting the solution, a message with the method used and if `A` was converted to the sparse form. For example:

[1] For Cholesky method we need a positive definite matrix. For the 5-Point FD Scheme the matrix `-A` computed only on the internal nodes of the grid has this property. In order to do that, build the matrix A for all the nodes with `build_linear_system`, then in `solve_linear_system` restrict the matrix `A` and the vector `b` only on the internal nodes, solve the system and add later the information about solution on the boundary points.

(a) The call `u= solve_linear_system(A,b,0,1,0)` has to display: "System solved with LU factorization in 3.2322 seconds."

(b) The call `u= solve_linear_system(A,b,1,0,0)` has to display: "System solved with `A\b` in 1.23212 seconds. The matrix A was converted in a sparse form."

Write your own `main91.m` and test all the different combinations[2] of "input answers" for the following problems[3]:

1. **Rectangular Domain:**

$$\Omega = [0,1] \times [0,1]$$
$$f(x_1,x_2) = -8\pi^2 \sin(2\pi x_1)\cos(2\pi x_2)$$
$$g(x_1,x_2) = \sin(2\pi x_1)\cos(2\pi x_2)$$

2. **Elliptic Domain:**

$$\Omega = \left\{ (x_1,x_2) \in \mathbb{R}^2,\ \frac{x_1^2}{4} + \frac{(x_2-5)^2}{9} \le 1 \right\}$$
$$f(x_1,x_2) = \widehat{f}(z) = \pi^2 \sin(2\pi z) - \frac{13}{9}\pi\cos(2\pi z) \quad \text{with } z = \frac{x_1^2}{4} + \frac{(x_2-5)^2}{9}$$
$$g(x_1,x_2) = 0$$

and plot the solutions with `plot_solution.m`. Write a report documenting how and why the different choices of `q1`,`q2` and `q3` affect the computational time for solving the linear system.

**Exercise 9.2** (Theory) (10 points)

Given the following Boundary Value Problem:

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = g & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_2 \end{cases} \tag{1}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^2$ with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $g \in C(\Gamma_1)$ and $f \in C(\overline{\Omega})$. Moreover, for a function $\psi \in C(\Gamma_1)$ let be

$$D_\psi = \left\{ u \in C(\overline{\Omega}) \,|\, u = \psi \text{ on } \Gamma_1 \right\},$$

and $\overline{u} \in C^2(\overline{\Omega}) \cap D_g$.

Prove the equivalence of these three following statements:

i) $\overline{u}$ solves the Boundary Value Problem (1)

ii) $\overline{u}$ is a stationary point of the functional $I : V_g \to \mathbb{R}$,

$$I(u) = \int_\Omega \left( \frac{1}{2}|\nabla u|^2 - fu \right) \mathrm{d}x\mathrm{d}y$$

where $V_\psi = H^1(\Omega) \cap \left\{ w \in C(\overline{\Omega}) \,|\, w = \psi \text{ on } \Gamma_1 \right\}$

iii) $u = \overline{u} \in V_g$ satisfies

$$\int_\Omega (\nabla u \cdot \nabla v - fv)\, \mathrm{d}x\mathrm{d}y = 0$$

for all $v \in V_0$

**Hints:**

1. For proving the equivalence "ii)$\Leftrightarrow$iii)" compute

$$\left. \frac{\partial}{\partial \varepsilon} I(\overline{u} + \varepsilon v) \right|_{\varepsilon=0}$$

2. For $v \in H^2(\Omega)$ and $w \in H^1(\Omega)$ holds:

$$\int_\Omega \nabla v \cdot \nabla w \mathrm{d}x\mathrm{d}y = -\int_\Omega \Delta v w \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega} \frac{\partial v}{\partial n} w dS,$$

where $n$ is the outward-pointing unit normal of $\Omega$. This generalized Green formula for $H^1$ function has not to be proved and can be used in the exercise.

[2]There are only 6 triplets (`q1`,`q2`,`q3`): `q2` and `q3` can not be 1 at the same time, because the user wants to solve the linear system only with one method.

[3]The 'Rectangular Domain' and 'Elliptic Domain' are defined in `main81.m` of Exercise 8.1.