Universität Konstanz

Fachbereich Mathematik
und Statistik

WS 2019/20

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Übungen zur **Mathematik I**

für die Studiengänge **Chemie, Life Science** und **Nanoscience**

Freiwillige Zusatzaufgaben zu **Anwendungen der Differentialrechnung**

Lösungen

(1) Es sei $p(x) = a_m x^m + \ldots + a_0$ ein Polynom vom Grad m und $\alpha > 0$.

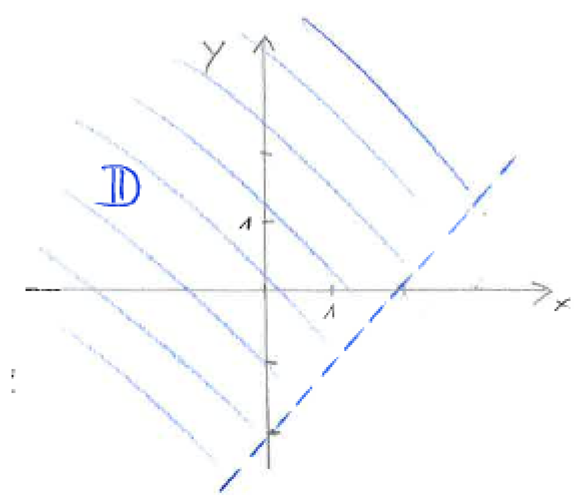
$$\lim_{x\to\infty} \frac{\exp(\alpha x)}{p(x)} = \begin{cases} +\infty & , \text{ falls } a_m > 0 \\ -\infty & , \text{ falls } a_m < 0 \end{cases} \quad , \qquad \lim_{x\to\infty} p(x)\cdot\exp(-\alpha x) = 0$$

$$\lim_{x\to\infty} \frac{\ln(x)}{p(x)} = 0 \quad , \qquad \lim_{x\to\infty} \frac{p(x)}{\ln(x)} = \begin{cases} +\infty & , \text{ falls } a_m > 0 \\ -\infty & , \text{ falls } a_m < 0 \end{cases}$$

(2)

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos(x)}{\pi - 2x} = \frac{1}{2} \quad ; \qquad \lim_{h\to 0} \frac{(2+h)^3 - (2-h)^3}{2h} = 12 \ .$$

(3) **a)** $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 \ : \ y > x - 2\}$



$\mathbb{W} = \mathbb{R}$

b)

$$h_x(x,y) = -\frac{3(x-1)^2}{(y+1)^3 - (x-1)^3}$$

$$h_y(x,y) = \frac{3(y+1)^2}{(y+1)^3 - (x-1)^3}$$

h besitzt keine lokalen Extrema, da es kein $(\bar{x},\bar{y})$ gibt mit $\nabla h(\bar{x},\bar{y}) = \vec{0}$.

(4)

$$h(x,y) = \ln\left((x+y)^2 + (x-y)^2\right) = \ln\left(x^2+2xy+y^2+x^2-2xy+y^2\right) = \ln\left(2x^2+2y^2\right)$$

a) $\mathbb{D} = \mathbb{R}^2 \backslash \{(0,0)\}$, $\mathbb{W} = \mathbb{R}$

b)

$$\begin{aligned}
h_x(x,y) &= \frac{4x}{2x^2+2y^2} = \frac{2x}{x^2+y^2}, \quad h_y(x,y) = \frac{4y}{2x^2+2y^2} = \frac{2y}{x^2+y^2} \\
\nabla h(x,y) &= \left(\frac{4x}{2x^2+2y^2}, \frac{4y}{2x^2+2y^2}\right) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right) \\
h_{xx}(x,y) &= \frac{4(2x^2+2y^2) - 4x \cdot 4x}{(2x^2+2y^2)^2} = \frac{8y^2-8x^2}{(2x^2+2y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} \\
h_{xy}(x,y) &= \frac{-4x \cdot 4y}{(2x^2+2y^2)^2} = \frac{-16xy}{(2x^2+2y^2)^2} = \frac{-4}{(x^2+y^2)^2} = h_{yx}(x,y) \\
h_{yy}(x,y) &= \frac{4(2x^2+2y^2) - 4y \cdot 4y}{(2x^2+2y^2)^2} = \frac{8x^2-8y^2}{(2x^2+2y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2} \\
\text{Hess}h(x,y) &= \begin{pmatrix} \frac{8y^2-8x^2}{(2x^2+2y^2)^2} & \frac{-16xy}{(2x^2+2y^2)^2} \\ \frac{-16xy}{(2x^2+2y^2)^2} & \frac{8x^2-8y^2}{(2x^2+2y^2)^2} \end{pmatrix} = \begin{pmatrix} \frac{2y^2-2x^2}{(x^2+y^2)^2} & \frac{-4}{(x^2+y^2)^2} \\ \frac{-4}{(x^2+y^2)^2} & \frac{2x^2-2y^2}{(x^2+y^2)^2} \end{pmatrix}
\end{aligned}$$

c) Für alle $(x,y) \in \mathbb{D}$ gilt $\nabla h(x,y) \neq 0$. Deshalb besitzt $h(x,y)$ keine lokale Extrema (da die notwendige Bedingung $\nabla h(x,y) = 0$ nie erfüllt wird).

d)

$$\begin{aligned}
(\overline{x}, \overline{y}) &= (1,2), \quad h(1,2) = \ln(10), \quad h_x(1,2) = \frac{2}{5}, \quad h_y(1,2) = \frac{4}{5} \\
h_{xx}(1,2) &= \frac{6}{25}, \quad h_{xy}(1,2) = -\frac{4}{25}, \quad h_{yy}(1,2) = -\frac{6}{25}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow p(x,y) &= h(\overline{x},\overline{y}) + h_x(\overline{x},\overline{y})(x-\overline{x}) + h_y(\overline{x},\overline{y})(y-\overline{y}) \\
&\quad + \frac{1}{2} h_{xx}(\overline{x},\overline{y})(x-\overline{x})^2 + h_{xy}(\overline{x},\overline{y})(x-\overline{x})(y-\overline{y}) + h_{yy}(\overline{x},\overline{y})(y-\overline{y})^2 \\
&= \ln(10) + \frac{2}{5}(x-1) + \frac{4}{5}(y-2) + \frac{3}{25}(x-1)^2 - \frac{4}{25}(x-1)(y-2) - \frac{3}{25}(y-2)^2
\end{aligned}$$