An optimal control problem for a parabolic PDE with control constraints

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The general optimal control problem

We consider the following convex optimization problem in Hilbert space:

$$\min_{(y,u)\in Y\times U} J(y,u) \qquad \text{subject to} \qquad E(y,u)=0,\ y\in Y_{\text{ad}},\ u\in U_{\text{ad}}$$

- Y state space, U control space (both Hilbert spaces)
- J convex cost functional
- E state equation (parabolic pde)
- $Y_{\text{ad}} \subseteq Y$ and $U_{\text{ad}} \subseteq U$ both nonempty, closed and convex, U_{ad} bounded.

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The cost functional

$$J(y,u) := \frac{1}{2}||y - y_Q||^2_{\mathcal{H}_Q} + \frac{1}{2}||y(T) - y_\Omega||^2_{\mathcal{H}_\Omega} + \frac{1}{2}||u - u_\Theta||^2_{\mathcal{H}_\Theta}$$

- $\Omega \subseteq \mathbb{R}^d$ open and bounded domain, $d \in \{1,2,3\}$
- $\Theta = (0,T)$ time interval, $T > 0$
- $Q = \Theta \times \Omega$ time-space cylinder
- $\mathcal{H}_Q := \mathcal{L}^2(Q),\ \mathcal{H}_\Omega := \mathcal{L}^2(\Omega),\ \mathcal{H}_\Theta := \mathcal{L}^2(\Theta)^m$
- $\langle\cdot,\cdot\rangle_{\mathcal{H}_Q} = \langle\sigma_Q\cdot,\cdot\rangle_{\mathcal{L}^2(Q)},\ \langle\cdot,\cdot\rangle_{\mathcal{H}_\Omega} = \langle\sigma_\Omega\cdot,\cdot\rangle_{\mathcal{L}^2(\Omega)},\ ,\ \langle\cdot,\cdot\rangle_{\mathcal{H}_\Theta} = \sum\langle\kappa_i\cdot,\cdot\rangle_{\mathcal{L}^2(\Theta)}$
- $\sigma_Q \in \mathcal{L}^\infty(Q),\ \sigma_\Omega \in \mathcal{L}^\infty(\Omega),\ \kappa_1,\ldots,\kappa_m \in \mathbb{R}$ nonnegative weights
- $y_Q \in \mathcal{L}^2(Q),\ y_\Omega \in \mathcal{L}^2(\Omega),\ u_\Theta \in \mathcal{L}^2(\Theta)^m$ (given) desired states and controls

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The state equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y(t,x) - \sigma \Delta y(t,x) = f(t,x) + \sum_{i=1}^{m} u_i(t)\chi_i(x) & \text{f.a.a. } (t,x) \in Q \\ y(t,x) = 0 & \text{f.a.a. } (t,x) \in \Theta \times \partial\Omega \\ y(0,x) = y_0(x) & \text{f.a.a. } x \in \Omega \end{cases}$$

- $f \in \mathcal{L}^2(Q),\ y_0 \in \mathcal{L}^2(\Omega),\ \sigma > 0,\ u_i \in \mathcal{L}^2(\Theta)$
- $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$ is the Gelfand triple $\mathcal{H}_0^1(\Omega) \hookrightarrow \mathcal{L}^2(\Omega) \hookrightarrow \mathcal{H}_0^1(\Omega)^*$
- $y \in Y := \{\varphi \in \mathcal{L}^2(\Theta, \mathcal{V}) \mid \varphi_t \in \mathcal{L}^2(\Theta, \mathcal{V}^*)\} \hookrightarrow \mathcal{C}^0(\Theta, \mathcal{L}^2(\Omega))$
- $Y_{\mathrm{ad}} := \{\varphi \in Y \mid y(0) = y_0\}$ is the space of admissible states
- $\chi_1, \ldots, \chi_m \in \mathcal{L}^2(\Omega)$ denote the control-shape functions

The control constraints

For the control functions $u_1, \dots, u_m$ we claim the box constraints

$$u_{a,i}(t) \leq u_i(t) \leq u_{b,i}(t) \qquad \text{f.a.a. } t \in \Theta,$$

i.e., the set of admissible control components is

$$U_{\text{ad}} := \{u \in \mathcal{L}^2(\Theta)^m \mid u \in [u_a, u_b] \text{ a.e. in } \Theta\}.$$

Notice that the set of admissible control-state pairs

$$X_{\text{ad}} = Y_{\text{ad}} \times U_{\text{ad}}$$

is nonempty, closed and convex.

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The Lagrange function

The corresponding *Lagrange functional*

$$\mathscr{L} : Y \times U \times \mathcal{L}^2(\Theta, \mathcal{V}) \to \mathbb{R}, \qquad (y, u, p) \mapsto \mathscr{L}(y, u, p)$$

is defined as

$$\mathscr{L}(y, u, p) = J(y, u) + E(y, u)p$$

$$= \frac{1}{2} \int_Q \sigma_Q |y - y_Q|^2 \, \mathrm{d}(x, t) + \frac{1}{2} \int_\Omega \sigma_\Omega |y(T) - y_\Omega|^2 \, \mathrm{d}x$$

$$+ \frac{1}{2} \sum_{i=1}^m \kappa_i \int_\Theta |u_i - u_{\Theta,i}|^2 \, \mathrm{d}t$$

$$+ \left\langle \left(\frac{\partial}{\partial t} - \sigma \Delta \right) y, p \right\rangle_{\mathcal{V}^*, \mathcal{V}} - \int_Q \left(f + \sum_{i=1}^m u_i \chi_i \right) p \, \mathrm{d}(x, t)$$

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Necessary first-order optimality conditions

The *variational inequalities*

$$\frac{\partial}{\partial y}\mathscr{L}(y^*,u^*,p^*)(y-y^*) \geq 0 \qquad \text{for all } y \in Y_{\text{ad}},$$
$$\frac{\partial}{\partial u_i}\mathscr{L}(y^*,u^*,p^*)(u_i-u_i^*) \geq 0 \qquad \text{for all } u_i \in U_{\text{ad}}.$$

hold in an optimal point $(y^*,u^*,p^*) \in X_{\text{ad}} \times \mathcal{L}^2(Q)$.

Additionally, the side-condition holds in (y^*,u^*,p^*):

$$\frac{\partial}{\partial p}\mathscr{L}(y^*,u^*,p^*) = E(y^*,u^*) = 0.$$

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The adjoint equation

The first variational inequality is equivalent to

$$\frac{\partial}{\partial y}\mathscr{L}(y^*,u^*,p^*)\tilde{y} = 0 \qquad \text{for all } \tilde{y} \in Y_0 := \{\varphi \in Y \mid \varphi(0) = 0\}.$$

By *calculus of variation*, we derive that p^* itself is a solution to a parabolic PDE:

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}p(t,x) - \sigma\Delta p(t,x) = -\sigma_Q(t,x)(y^*(t,x) - y_Q(t,x)) & \text{a.e. in } Q \\ p(t,x) = 0 & \text{a.e. in } \Theta \times \partial\Omega \\ p(T,x) = -\sigma_\Omega(x)(y^*(T,x) - y_\Omega(x)) & \text{a.e. in } \Omega \end{cases}$$

Since the solution operator to this backwards heat equation is the adjoint solution operator to the state equation, the Lagrange multiplier p^* is called the *adjoint state*.

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The projection condition

The second variational inequality is equivalent to

$$-\sum_{i=1}^{m}\int_{Q}\chi_i p^*\tilde{u}_i\,\mathrm{d}(x,t)+\sum_{i=1}^{m}\kappa_i\int_{\Theta}(u_i^*-u_{\Theta,i})\tilde{u}_i\,\mathrm{d}t\Big|_{\tilde{u}_i=u_i-u_i^*}\geq 0.$$

Hence, the components of u^* fulfill the projection conditions

$$u_i^*(t)=\mathbb{P}_{[u_{a,i}(t),u_{b,i}(t)]}\left(\frac{1}{\kappa_i}\int_{\Omega}\chi_i(x)p^*(t,x)\,\mathrm{d}x+u_{\Theta,i}(t)\right).$$

Since $p^*=S^*y^*$ and $y^*=Su^*$ hold, u^* can be determined as the single *fixpoint* of

$$\boxed{u_i\mapsto\mathbb{P}_{[u_{a,i},u_{b,i}]}\left(\frac{1}{\kappa_i}\int_{\Omega}\chi_i(x)(S^*Su)(x)\,\mathrm{d}x+u_{\Theta,i}\right)}$$

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Existence and uniqueness

Let $S : u \mapsto y(u)$ the solution operator for the state equation, i.e.

$$E(y,u) = 0 \qquad \Longleftrightarrow \qquad y = Su.$$

The optimization problem then can be replaced by the *reduced problem*

$$\min_{u \in U_{\text{ad}}} \mathscr{F}(u) := \frac{1}{2}||Su - y_Q||^2_{\mathcal{H}_Q} + \frac{1}{2}||(Su)(T) - y_\Omega||^2_{\mathcal{H}_\Omega} + \frac{1}{2}||u - u_\Theta||^2_{\mathcal{H}_\Theta}$$

Since S is affine linear and continuous and U_{ad} is nonempty, closed and convex, this quadratic optimization problem has a solution.

Under the assumption that the *regularization parameters* $\kappa_1, ..., \kappa_m$ are strictly positive, this solution is unique.

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Projected gradient method (PGM)

ALGORITHM: Let $u_n \in U_{\text{ad}}$ any non-optimal admissible control.

Task: Find a *direction* $d_n \in \mathcal{L}^2(\Theta)^m$, $||d_n||_{\mathcal{L}^2(\Theta)^m} = 1$, and a *stepsize* $s_n > 0$ such that

$$u_{n+1} := \mathbb{P}_{[u_a,u_b]}(u_n + s_n d_n) \ll u_n.$$

In this context, $\ll$ means that $(u_n)_{n\in\mathbb{N}}$ converges at least linearly towards u^*.

We choose d_n as the *negative gradient* $d_n := -\nabla\mathscr{F}(u_n)$ (Taylor & Cauchy-Schwarz).

An efficient stepsize is $s_n(N^*) := \frac{1}{2^{N^*}}$ where N^* is the minimal $N \in \mathbb{N}$ such that the ARMIJO *condition* holds:

$$\mathscr{F}(\mathbb{P}_{[u_a,u_b]}(u_n + s(N)d_n)) - \mathscr{F}(u_n) \leq -\frac{1}{2s(N)}||\mathbb{P}_{[u_a,u_b]}(u_n + s(N)d_n) - u_n||_{\mathcal{L}^2(\Theta)^m}.$$

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Primal-dual active set strategy

The projection condition

$$u_i^* = \mathbb{P}_{[u_{a,i},u_{b,i}]}\left(\frac{1}{\kappa_i}\int\limits_\Omega \chi_i(x)p^*(x)\,\mathrm{d}x + u_{\Theta,i}\right)$$

can be written as

$$\boxed{u_i^* = (1-\chi_{A_i}-\chi_{B_i})\left(\frac{1}{\kappa_i}\int\limits_\Omega \chi_i(x)p^*(x)\,\mathrm{d}x + u_{\Theta,i}\right) + \chi_{A_i}u_{a,i} + \chi_{B_i}u_{b,i}}$$

where χ_M denotes the indicator function

$$\chi_M(x) := \begin{cases} 1 & x \in M \\ 0 & x \notin M \end{cases}$$

and the *active sets* A_i, B_i are

$$A_i := \left\{ t \in \Theta \;\middle|\; \frac{1}{\kappa_i}\int\limits_\Omega \chi_i(x)p^*(t,x)\,\mathrm{d}x + u_{\Theta,i}(t) < u_{a,i}(t)\right\};$$

$$B_i := \left\{ t \in \Theta \;\middle|\; \frac{1}{\kappa_i}\int\limits_\Omega \chi_i(x)p^*(t,x)\,\mathrm{d}x + u_{\Theta,i}(t) > u_{b,i}(t)\right\}.$$

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Primal-dual active set strategy

The optimality system then is given as the semilinear coupled parabolic system

$$\begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}t} - \sigma\Delta\right) y - \sum_{i=1}^{m} \chi_i u_i = f \\ \left(-\frac{\mathrm{d}}{\mathrm{d}t} - \sigma\Delta\right) p + \sigma_Q y = \sigma_Q y_Q \\ (1 - \chi_{A_i} - \chi_{B_i}) \left(-\frac{1}{\kappa_i} \int_\Omega \chi_i p \,\mathrm{d}x\right) + u_i = (1 - \chi_{A_i} - \chi_{B_i}) u_{\Theta,i} + \chi_{A_i} u_{a,i} + \chi_{B_i} u_{b,i} \end{cases}$$

ALGORITHM: Let (p_n, y_n, u_n) given.

1. $A_i^{n+1} := \left\{ t \in \Theta \mid \frac{1}{\kappa_i} \int_\Omega \chi_i(x) p_n(t,x) \,\mathrm{d}x + u_{\Theta,i}(t) < u_{a,i}(t) \right\};$

   $B_i^{n+1} := \left\{ t \in \Theta \mid \frac{1}{\kappa_i} \int_\Omega \chi_i(x) p_n(t,x) \,\mathrm{d}x + u_{\Theta,i}(t) > u_{b,i}(t) \right\};$
2. **if** $A_i^{n+1} = A_i^n$ **and** $B_i^{n+1} = B_i^n$ **finish**
   else $(p_{n+1}, y_{n+1}, u_{n+1}) :=$ the solution of the pde system
3. $n := n + 1$

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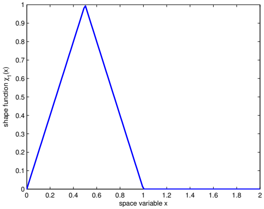
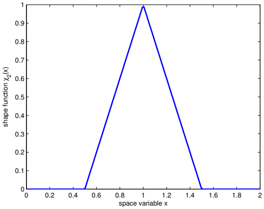
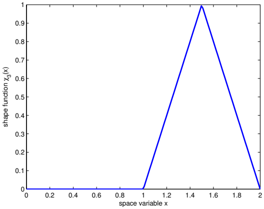
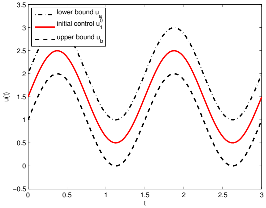
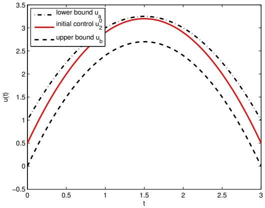
Shape functions, initial control and control constraints

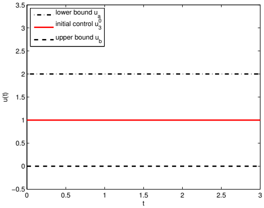












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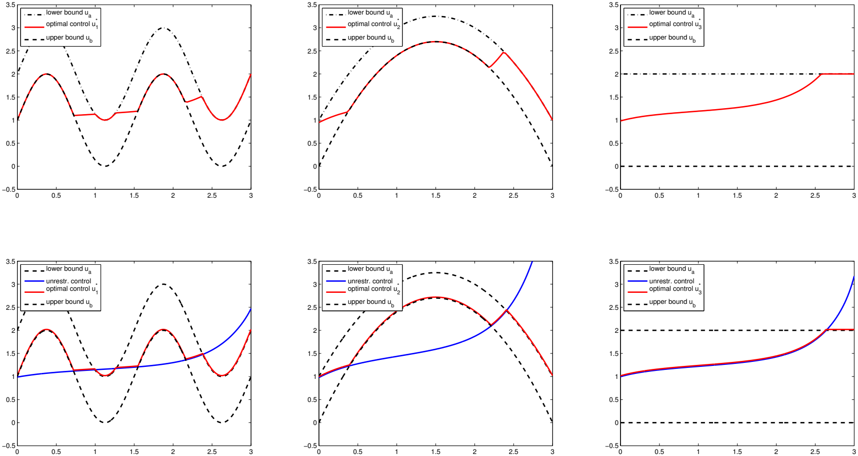
Optimal control components with and without restrictions



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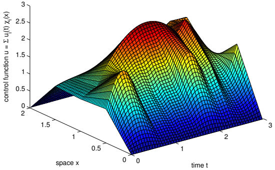
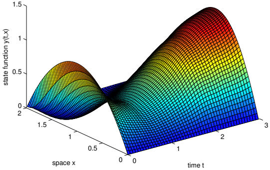
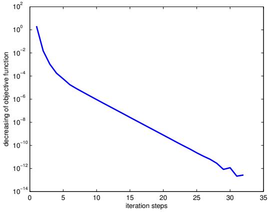
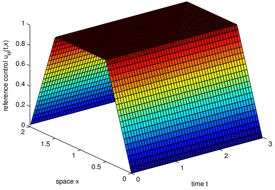
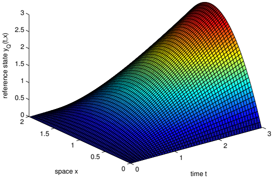
Optimal and desired control and state

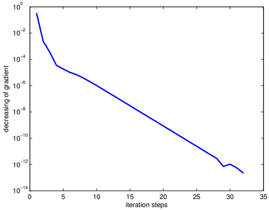












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The reduced order model

In the following, we consider the non-discretized, unconstraint optimization problem

$$\min_{y,u} J(y,u) := \frac{\sigma_Q}{2} \int_Q |y - y_Q|^2 \, \mathrm{d}(x,t) + \frac{1}{2} \sum_{i=1}^{m} \kappa_i \int_\Theta |u_i(t)|^2 \, \mathrm{d}t$$

subject to

$$\left\{ \begin{array}{rcl} \dot{y}(t) + Ay(t) & = & f(t) + Bu(t) \\ y(0) & = & y_0 \end{array} \right. \tag{P}$$

with $A = -\sigma\Delta$ and $(Bu)(t) := \sum_{i=1}^{m} \chi_i(x) u_i(t)$.

A POD basis $\psi = (\psi_1, ..., \psi_l)$ for y is given by the eigenvectors belonging to the l largest eigenvalues of

$$\mathcal{Y}\mathcal{Y}^* : \mathcal{H} \to \mathcal{H}, \qquad (\mathcal{Y}\mathcal{Y}^*)\psi = \int_\Theta \langle y(t), \psi \rangle_{\mathcal{H}} y(t) \, \mathrm{d}t.$$

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The reduced order model

The reduced model reads as

$$\min_{\mathrm{y},u} \mathrm{J}(\psi)(\mathrm{y},u) = \frac{\sigma_Q}{2} \int_\Theta \langle \mathrm{y}(t), \mathrm{M}(\psi)\mathrm{y}(t)\rangle - 2\langle \mathrm{y}(t), \mathrm{y}_Q(t)\rangle + ||\mathrm{y}_Q(t)||^2 \, \mathrm{d}t$$

$$+ \sum_{i=1}^{m} \kappa_i \int_\Theta |u_i(t)|^2 \, \mathrm{d}t$$

subject to

$$\left\{ \begin{array}{rcl} \mathrm{M}(\psi)\dot{\mathrm{y}}(t) + \mathrm{A}(\psi)\mathrm{y}(t) & = & \mathrm{f}(\psi)(t) + \mathrm{B}(\psi)u(t) \\ \mathrm{M}(\psi)\mathrm{y}(0) & = & \mathrm{y}_0(\psi) \end{array} \right. \qquad (\mathrm{P}^l)$$

where

$$\begin{array}{rclrcl} \mathrm{M}(\psi) & = & (\langle \psi_i, \psi_j \rangle_{\mathcal{H}}) \in \mathbb{R}^{l\times l}, & \mathrm{A}(\psi) & = & (\langle \psi_i, A\psi_j \rangle_{\mathcal{H}}) \in \mathbb{R}^{l\times l}, \\ \mathrm{f}(\psi) & = & (\langle f, \psi_i \rangle_{\mathcal{H}}) \in \mathcal{L}^2(\Theta, \mathbb{R}^{l\times 1}), & \mathrm{B}(\psi) & = & (\langle \chi_i, \psi_j \rangle_{\mathcal{H}}) \in \mathbb{R}^{l\times m}, \\ \mathrm{y}_Q(\psi) & = & (\langle y_\Omega, \psi_i \rangle_{\mathcal{H}}) \in \mathcal{L}^2(\Theta, \mathbb{R}^{l\times l}), & \mathrm{y}_0(\psi) & = & (\langle y_0, \psi_i \rangle_{\mathcal{H}}) \in \mathbb{R}^{l\times 1}. \end{array}$$

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Implementation for (P^l)

ALGORITHM: Let u_n an admissible control.

1. Solve the full PDE $E(y, u_n) = 0$ to get the corresponding state y^n.
2. Compute a POD basis (ψ_n^l) for y^n.
3. Solve (P^l) with a globalized SQP method (Sequential Quadratic Programming, a Newton method) to get a better control u_{n+1}.
4. $n := n + 1$

- Since the expensive optimization is done for the small system, this approach needs less effort than solving (P).
- Disadvantage: If the dynamics of y_n differ much from those of the optimal state y^*, the POD basis (ψ_n^l) is not very efficient.
- This weakness can be eliminated by choosing (ψ_l^n) in a way such that the goal of minimizing J is respected.

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The POD optimality system

$$\min_{\mathrm{y},u,\psi} \mathrm{J}(\mathrm{y},u,\psi) := \mathrm{J}(\psi)(\mathrm{y},u)$$

subject to

$$\left\{\begin{array}{rcl} \mathrm{E}(\psi)(\mathrm{y},u) = \mathrm{M}(\psi)\dot{\mathrm{y}}(t) + \mathrm{A}(\psi)\mathrm{y}(t) - \mathrm{f}(\psi)(t) - \mathrm{B}(\psi)u(t) & = & 0 \\ \mathrm{M}(\psi)\mathrm{y}(0) & = & \mathrm{y}_0(\psi) \\ & & \\ E(y,u) = \dot{y}(t) + Ay(t) - f(t) - Bu(t) & = & 0 \\ y(0) & = & y_0 \\ & & \\ \mathcal{Y}\mathcal{Y}^*\psi_i & = & \lambda_i\psi_i \\ \langle\psi_i,\psi_j\rangle_{\mathcal{H}} & = & \delta_{ij} \end{array}\right. \qquad (\mathrm{P}^l_{\text{OS-POD}})$$

To derive necessary first-order conditions, we use the Lagrange framework. Let

$$\begin{array}{lclcl} z & := & (\mathrm{y},y,\psi,\lambda,u) & \in Z & := \mathcal{H}^1(\Theta,\mathbb{R}^l) \times \mathcal{W}(\Theta) \times \Psi^l \times \mathbb{R}^l \times \mathcal{L}^2(\Theta,\mathbb{R}^m) \\ \xi & := & (q,p,\mu,\eta) & \in \Xi & := \mathcal{L}^2(\Theta,\mathbb{R}^l) \times \mathcal{L}^2(\Theta,\mathcal{V}) \times \Psi^l \times \mathbb{R}^l. \end{array}$$

Lagrange functional & necessary first-order conditions

Define the Lagrange functional $\mathscr{L} : Z \times \Xi \to \mathbb{R}$ as

$$\mathscr{L}(z,\xi) = \mathrm{J}(\mathrm{y},u,\psi) + \langle \mathrm{E}(\psi)(\mathrm{y},u), q\rangle_{\mathcal{L}^2(\Theta,\mathbb{R}^m)} + \langle E(y,u), p\rangle_{\mathcal{L}^2(\Theta,\mathcal{V}^*),\mathcal{L}^2(\Theta,\mathcal{V})}$$
$$+ \sum_{i=1}^{l} \langle (\mathcal{Y}\mathcal{Y}^* - \lambda_i \mathrm{Id})\psi_i, \mu_i\rangle_{\Psi^l} + \sum_{i=1}^{l} \langle ||\psi_i||^2_{\Psi^l} - 1, \eta_i\rangle_{\mathbb{R}^l}.$$

The necessary first-order conditions for an optimal point (z,ξ) are:

- $\mathscr{L}_{\mathrm{y}}(z,\xi) = 0$ which implies

$$-\mathrm{M}(\psi)\dot{q}(t) + \mathrm{A}(\psi)q(t) = -\sigma_Q(\mathrm{E}(\psi)\mathrm{y}(t) - \mathrm{y}_Q(\psi)(t)) + \text{final condition;}$$

- $\mathscr{L}_{y}(z,\xi) = 0$ which implies

$$-\dot{p}(t) + Ap(t) = \sum_{i=1}^{l} \langle y(t), \mu_i\rangle_{\mathcal{H}}\psi_i + \langle y(t), \psi_i\rangle_{\mathcal{H}}\mu_i + \text{final condition;}$$

Lagrange functional & necessary first-order conditions

- $\mathscr{L}_\lambda(z,\xi) = 0$ which implies the "complementary slackness condition"

$$\langle \psi_i, \mu_i \rangle_{\mathcal{H}} = 0;$$

- $\mathscr{L}_\psi(z,\xi) = 0$ which implies

$$\begin{aligned}
0 &= \sum_{i=1}^{l} \langle (\mathcal{Y}\mathcal{Y}^* - \lambda_i \mathrm{Id})\cdot, \mu_i \rangle_{\mathcal{H}} + 2 \sum_{i=1}^{l} \langle \cdot, \psi_i \rangle_{\mathcal{H}} \eta_i \\
&\quad + \sigma_Q \int_\Theta \langle \mathrm{y}(t), \mathrm{M}(\psi, \cdot)\mathrm{y(t)} - \mathrm{y}_Q(t, \cdot) \rangle_{\mathbb{R}^m} \, \mathrm{d}t \\
&\quad + \int_\Theta \langle q(t), (\mathrm{M}(\psi, \cdot) + \mathrm{M}(\cdot, \psi))\dot{\mathrm{y}}(t) + (\mathrm{A}(\psi, \cdot) + \mathrm{A}(\cdot, \psi))\mathrm{y}(t) \\
&\qquad\qquad - \mathrm{B}(\cdot) u(t) \rangle_{\mathbb{R}^m} \, \mathrm{d}t \\
&=: \sum_{i=1}^{l} \langle (\mathcal{Y}\mathcal{Y}^* - \lambda_i \mathrm{Id})\cdot, \mu_i \rangle_{\mathcal{H}} + 2 \langle \cdot, \psi_i \rangle_{\mathcal{H}} \eta_i + \langle g_i(\mathrm{y}, \psi, u, q), \cdot \rangle_{\mathcal{H}^*, \mathcal{H}}.
\end{aligned}$$

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Lagrange functional & necessary first-order conditions

- Hereby, we set $\mathrm{M}(\psi,\phi) := (\langle\psi_i,\phi_j\rangle_{\mathcal{H}})$, $\mathrm{A}(\psi,\phi) = (\langle\psi_i, A\phi_j\rangle_{\mathcal{V}*,\mathcal{V}})$ and $\mathrm{B}(\phi) = (\langle\chi_i,\phi_i\rangle)_{\mathcal{H}}$.

  Together with the complementary slackness, this implies

$$\eta_i = -\frac{1}{2}\langle g_i(\mathrm{y},\psi,u,q),\psi\rangle_{\mathcal{H}^*,\mathcal{H}};$$
$$(\mathcal{Y}\mathcal{Y}^* - \lambda_i \mathrm{Id})\mu_i = -2\eta_i\psi_i - g_i(\mathrm{y},\psi,u,q).$$

- Finally, $\mathscr{L}_u(z,\xi) = 0$ which implies

$$\sum_{i=1}^{m}\kappa_i u_i(t) = \mathrm{B}^t(\psi)q(t) + B^* p(t);$$

  in case of *constraint* optimization, here the fun begins.

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Optimality system

We do not solve the single equations in the resulting optimality system

$$\begin{cases} -\mathrm{M}(\psi)\dot{q}(t) + \mathrm{A}(\psi)q(t) &= -\sigma_Q(\mathrm{M}(\psi)\mathrm{y}(t) - \mathrm{y}_Q(\psi)(t)) + \text{final cond.} \\ -\dot{p}(t) + Ap(t) &= \sum\langle y(t), \mu_i\rangle_{\mathcal{H}}\psi_i + \langle y(t), \psi_i\rangle_{\mathcal{H}}\mu_i + \text{final cond.} \\ \eta_i &= -\frac{1}{2}(g_i(\mathrm{y}, \psi, u, q), q)_{\mathcal{H}^*,\mathcal{H}} \\ \mu_i &= -(\mathcal{Y}\mathcal{Y}^* - \lambda_i \mathrm{Id})^{-1}(2\eta_i\psi_i + g_i(\mathrm{y}, \psi, u, q)) \\ \sum \kappa_i u_i(t) &= \mathrm{B}^t(\psi)q(t) + B^* p(t), \end{cases}$$

simultaniously, but use the operator splitting ansatz $z_1 = z_1(\mathrm{y}, \psi, u) := (y^l, u)$ and $z_2 = (y, \psi, \lambda)$.

- The optimization subproblem (P^l) within ($\mathrm{P}^l_{\text{OS-POD}}$) determines $z_1^* = z_1^*(z_2)$ for fixed z_2 with SQP.
- The remaining minimization with respect to z_2 is done with the Gradient Method; this requires the expensive solutions of the full system. Hereby, the Gradient Method requires one forward and one backward solve of $E(y, u) = 0$ and the update of the POD basis requires one forward solve.
- In case of constraints, this step becomes very problematic as we have seen before.

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Splitting optimization algorithm

$$\begin{cases} -\mathrm{M}(\psi)\dot{q}(t) + \mathrm{A}(\psi)q(t) &= -\sigma_Q(\mathrm{M}(\psi)\mathrm{y}(t) - \mathrm{y}_Q(\psi)(t)) + \text{final cond.} \\ -\dot{p}(t) + Ap(t) &= \sum \langle y(t), \mu_i \rangle_{\mathcal{H}} \psi_i + \langle y(t), \psi_i \rangle_{\mathcal{H}} \mu_i + \text{final cond.} \\ \eta_i &= -\frac{1}{2}(g_i(\mathrm{y}, \psi, u, q), q)_{\mathcal{H}^*, \mathcal{H}} \\ \mu_i &= -(\mathcal{Y}\mathcal{Y}^* - \lambda_i \mathrm{Id})^{-1}(2\eta_i\psi_i + g_i(\mathrm{y}, \psi, u, q)) \\ \sum \kappa_i u_i(t) &= \mathrm{B}^t(\psi)q(t) + B^* p(t), \end{cases}$$

ALGORITHM: Let a POD basis ψ^n given.

1. Solve (P^l): $\min \mathrm{J}(\psi^n)(\mathrm{y}^{n+1}, \tilde{u}^{n+1})$ s.t. $\mathrm{E}(\psi)(\mathrm{y}^{n+1}, \tilde{u}^{n+1}) = 0,\ \mathrm{y}^{n+1}(0) = \mathrm{y}_0(\psi^n)$ by SQP. Hereby, q^{n+1} is determined, too.
2. Solve $E(y^{n+1}, \tilde{u}^{n+1}) = 0,\ y(0) = y_0$, and calculate $\eta_i^{n+1},\ \mu_i^{n+1}$ to determine p^{n+1}.
3. Choose direction $-\nabla\mathscr{F}(u_n) = (\frac{1}{\kappa_i}\mathrm{B}^t(\psi_i^n)q^{n+1}(t) + B^* p^{n+1}(t))$ in a gradient step to get a new control u^{n+1}.
4. Determine a new POD basis ψ^{n+1} by solving $E(y^{n+1}, u^{n+1}) = 0,\ y(0) = y_0$ and set $n := n + 1$.

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The next steps

We postulate additional mixed control-state box constraints

$$a \leq \sum_{i=1}^{l} \mathrm{y}_i(t)\psi_i + \epsilon \sum_{i=1}^{m} u_i(t)\chi_i \leq b$$

To-Do-List:

- Existence of optimal solutions to $(\mathrm{P}^l_{\text{OS-POD}})$?
- Existence of Lagrange multipliers?
- A-posteriori error estimates?
- Behavior for $\epsilon \to 0$ (pure state constraints)?

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