# Proper Orthogonal Decomposition for Optimal Control Problems with Mixed Control-State Constraints

### Workshop on Numerical Methods for Optimal Control and Inverse Problems

### Technische Universität München

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# Problem formulation

$$\min_{(y,u)\in Y\times U} J(y,u) = \frac{1}{2}||y - y_Q||^2_{L^2(0,T;H)} + \frac{\kappa}{2}||u||^2_U \qquad (*)$$

subject to

$$\langle \dot{y}(t), \varphi \rangle_{V^\star,V} + \langle \nabla y(t), \nabla \varphi \rangle_H = \langle (\mathcal{B}u + f)(t), \varphi \rangle_{V^\star,V} \qquad (\varphi \in V = H^1_0(\Omega))$$

$$\langle y(0), \varphi \rangle_H = \langle y_\circ, \varphi \rangle_H \qquad (\varphi \in H = L^2(\Omega))$$

and

$$\varepsilon u(t) + (\mathcal{I}y)(t) \leq u_b(t)$$

with $U = L^2(0, T; \mathbb{R}^m)$ and $\mathcal{B} : U \to L^2(0, T; H), \mathcal{I} : L^2(0, T; H) \to U$ given by

$$(\mathcal{B}u)(t,x) = \sum_{i=1}^m u_i(t)\chi_i(x), \qquad (\mathcal{I}y)(t) = \left( \int_{\text{supp}(\chi_i)} y(t,x)\mathrm{d}x \right).$$

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# Existence and uniqueness

**THEOREM.** Assume that the closed, convex and bounded set

$$X_{\mathrm{ad}} = \{(y, u) \in Y \times U \mid \dot{y} - \Delta y = \mathcal{B}u + f \ \& \ y(0) = y_\circ \ \& \ \varepsilon u + \mathcal{I}y \le u_b\}$$

is nonempty. Then there exists a unique solution $(\bar{y}, \bar{u})$ to $(*)$.

Further, there is a unique pair of Lagrange multipliers $(\bar{p}, \bar{\lambda})$ such that

$$\dot{\bar{y}}(t) + \langle \nabla \bar{y}(t), \nabla \cdot \rangle_H - (\mathcal{B}\bar{u} + f)(t) = 0$$

$$y(0) - y_\circ = 0$$

$$-\dot{\bar{p}}(t) + \langle \nabla \bar{p}(t), \nabla \cdot \rangle_H + (\mathcal{I}^\star \bar{\lambda})(t) + (\bar{y} - y_Q)(t) = 0$$

$$\bar{p}(T) = 0 \qquad\qquad (**)$$

$$\kappa \bar{u}(t) - (\mathcal{B}^\star p)(t) + \varepsilon \bar{\lambda}(t) = 0$$

$$\bar{\lambda}(t) - \chi_{\mathcal{A}_b(y,p)}\left(\frac{1}{\varepsilon}\mathcal{B}^\star p + \frac{\kappa}{\varepsilon}\mathcal{I}y - \frac{\kappa}{\varepsilon^2}u_b\right) = 0$$

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# Primal-dual active set strategy

**Algorithm** (Primal-dual active set strategy)

**Require:** Initial state-adjoint state pair $(y^0, p^0) \in Y \times Y$.

  1: Set $k = 0$

  2: **repeat**

  3:      Calculate active set $\mathcal{A}_b(y^k, p^k)$

  4:      Solve the primal-dual system $(**)$ to get $(y^{k+1}, p^{k+1})$

  5:      Set $k = k + 1$

  6: **until** $\mathcal{A}_b(y^k, p^k) = \mathcal{A}_b(y^{k-1}, p^{k-1})$

  7: Return $\bar{u} = u(y^k, p^k)$.

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# Proper orthogonal decomposition (POD)

**Problem:** After elimination of $u, \lambda$, the discrete linear system $(**)$ is still of the dimension $2N_t N_x$.

**Idea:** For $\ell \ll N_x$, find an *optimal* finite element basis $(\psi_1, ..., \psi_\ell)$ such that the corresponding Galerkin solution $y^\ell$ is close to $\bar{y}$.

**THEOREM.** Let $y_{\text{FE}} \in \mathcal{C}^1(0, T; \mathbb{R}^{N_x})$ an FE solution to the state equation. Consider the operators $\mathcal{Y} : L^2(0, T; \mathbb{R}) \to \mathbb{R}^{N_x}$ and $\mathcal{Y}^\star : \mathbb{R}^{N_x} \to L^2(0, T; \mathbb{R}))$,

$$\mathcal{Y}\phi = \langle \phi, y_{\text{FE}} \rangle_{L^2(0,T;\mathbb{R})} \in \mathbb{R}^{N_x}, \qquad \mathcal{Y}^\star\phi = \langle \phi, y_{\text{FE}}(\cdot) \rangle_H \in L^2(0, T; \mathbb{R}).$$

Let $\psi^\ell = (\psi_1, ..., \psi_\ell)$ eigenvectors to $\mathcal{Y}\mathcal{Y}^\star : \mathbb{R}^{N_x} \to \mathbb{R}^{N_x}$ corresponding to the $\ell$ largest eigenvalues. Then $\psi^\ell$ solves the minimization problem

$$\min_{\substack{i=1,...,\ell \\ \langle \phi_i, \phi_j \rangle_H = \delta_{ij}}} \left\| y_{\text{FE}} - \underbrace{\sum_{i=1}^{\ell} \langle y_{\text{FE}}, \phi_i \rangle_H \phi_i}_{=y^\ell} \right\|_H^2.$$

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# Reduced order modeling (ROM)

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**Algorithm**  (Reduced order modeling)

**Require:** $\ell \ll N_x$, initial control guess $u^\ell \in U$, desired exactness $\epsilon$

1: **repeat**
2:    Calculate full-order state solution $y_{\text{FE}} = y(u^\ell)$
3:    Calculate POD basis $\psi = \psi(y_{\text{FE}})$
4:    Apply PDASS on the reduced system $(**)$ to get $u^\ell = u(\psi)$
5: **until** $\text{Aposti}(u^\ell) < \epsilon$
6: Return $\bar{u} = u^\ell$

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# Numerical example

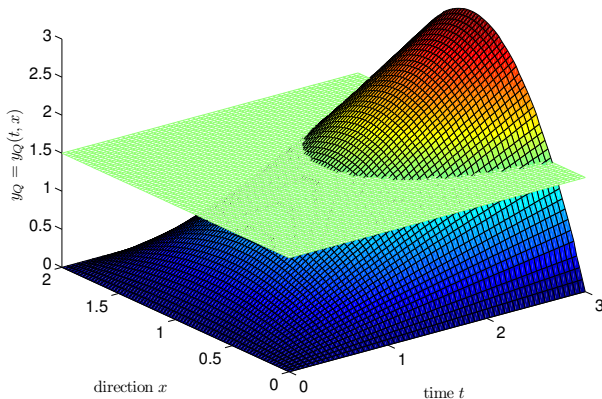desired state $y_Q$ & state constraint $u_b$



**Fig. 1.** The desired state $y_Q \in L^2(0, T; H)$ and the upper mixed control-state bound $u_b$, interpreted as a pure pointwise state constraint.
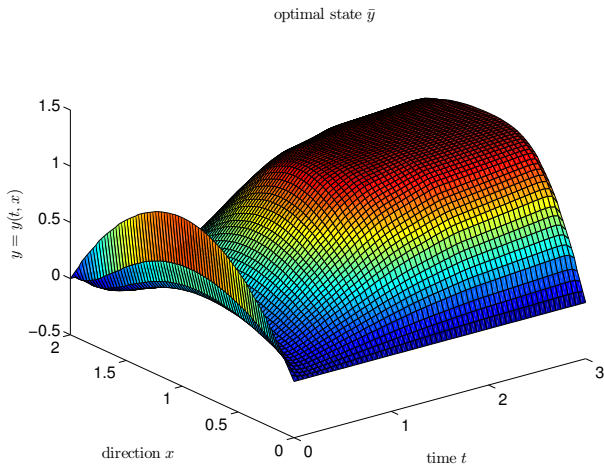
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# Numerical example

optimal state $\bar{y}$



**Fig. 2.** The optimal state solution $\bar{y} \in Y$ satisfying the state constraints.

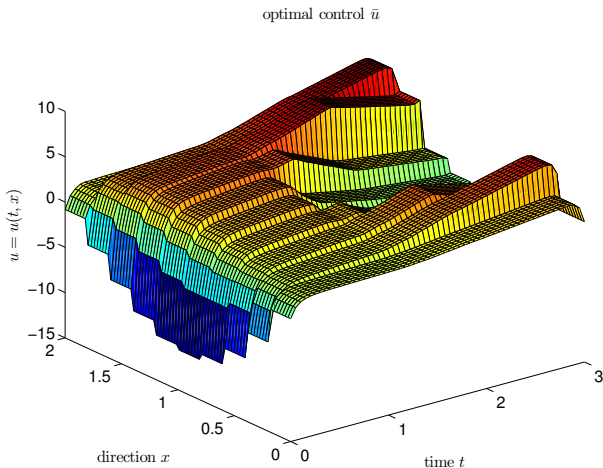Universität
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# Numerical example



**Fig. 3.** The optimal control term $\mathcal{B}\bar{u} \in L^2(0, T; H)$ controlling the state equation.
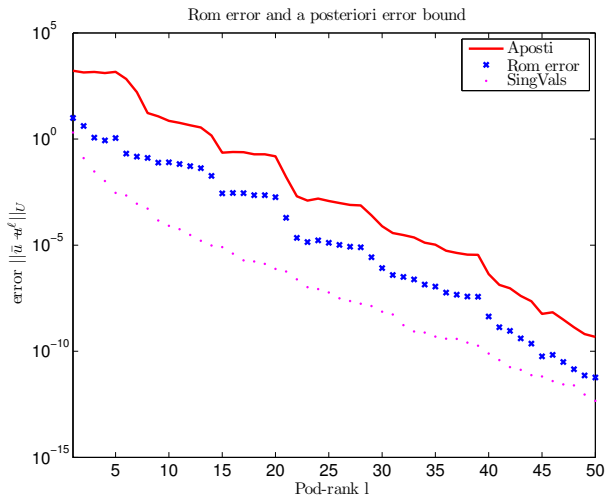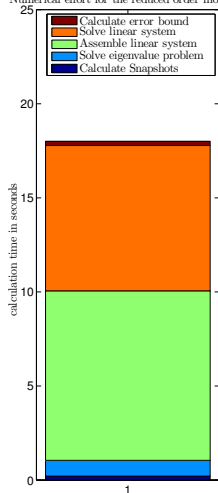
# Numerical results



**Fig. 4.** The ROM errors of the control $u^\ell$ for different POD basis ranks $\ell$, the error bound provided by the a posteriori error estimator and the singular values (squares of the eigenvalues) of the operator $\mathcal{Y}\mathcal{Y}^\star$ which can be used as a costless error indicator.

# Numerical effort



Numerical effort for the reduced order model

| Process | Time | # | Total |
|---------|------|---|-------|
| Assemble full system | 0.66 sec | 9× | 5.97 sec |
| Solve full system | 22.27 sec | 9× | 200.43 sec |
| **Total** | | | 206.40 sec |
| | | | |
| Solve full snapshots equations | 0.11 sec | 2× | 0.21 sec |
| Solve eigenvalue problem | 0.42 sec | 2× | 0.84 sec |
| Assemble ROM system | 0.53 sec | 17× | 9.01 sec |
| Solve ROM system | 0.45 sec | 17× | 7.72 sec |
| Evaluate error estimator | 0.11 sec | 2× | 0.23 sec |
| **Total** | | | 18.01 sec |

**Tab. 1.** The calculation times for solving the optimization problem with and without model reduction. With 25 POD elements, the ROM problem has to be solved two times; solvings of two eigenvalue problems are required in addition to update the POD basis. Nevertheless, 91.27% of the calculation time is spared in total.

**Fig. 5.** The numerical effort of the single algorithm processes for solving the ROM problem on an Intel(R) Core(TM) i5 2.40GHz processor.

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