Proper Orthogonal Decomposition for Optimal Control Problems with Mixed Control-State Constraints

Technische Universität Berlin

Martin Gubisch, Stefan Volkwein

University of Konstanz

March 20, 2013

Universität Konstanz

Problem formulation

$$\min_{(y,u)\in Y\times U} J(y,u) = \frac{1}{2}||y - y_Q||^2_{L^2(0,T;H)} + \frac{\kappa}{2}||u||^2_U \tag{OCP}$$

subject to

$$\langle \dot{y}(t), \varphi\rangle_{V^\star,V} + \langle \nabla y(t), \nabla\varphi\rangle_H = \langle (\mathcal{B}u + f)(t), \varphi\rangle_{V^\star,V} \qquad (\varphi \in V = H^1_0(\Omega))$$
$$\langle y(0), \varphi\rangle_H = \langle y_\circ, \varphi\rangle_H \qquad (\varphi \in H = L^2(\Omega))$$

and

$$\varepsilon u(t) + (\mathcal{I}y)(t) \le u_b(t)$$

with $U = L^2(0,T;\mathbb{R}^m)$ and $\mathcal{B} : U \to L^2(0,T;H), \mathcal{I} : L^2(0,T;H) \to U$ given by

$$(\mathcal{B}u)(t,x) = \sum_{i=1}^m u_i(t)\chi_i(x), \qquad (\mathcal{I}y)(t) = \left(\int_{\mathrm{supp}(\chi_i)} y(t,x)\mathrm{d}x\right).$$

Universität Konstanz

Lagrange functional

Define the Lagrange functional

$$\mathscr{L} : Y \times U \times L^2(0,T;V) \times H \times L^2(0,T;\mathbb{R}^m)$$

by

$$\begin{aligned}\mathscr{L}(y,u,p,\lambda) = J(y,u) &+ \langle E(y,u), p\rangle_{L^2(0,T;V^\star),L^2(0,T;V)} \\ &+ \langle y(0) - y_\circ, p_\circ\rangle_H \\ &+ \langle (\mathcal{I}y + \varepsilon u) - u_b, \lambda\rangle_{L^2(0,T;\mathbb{R}^m)}\end{aligned}$$

where

$$(E(y,u)\varphi)(t) = \langle \dot{y}(t), \varphi\rangle_{V^\star,V} + \langle \nabla y(t), \nabla\varphi\rangle_H - \langle (\mathcal{B}u + f)(t), \varphi\rangle_{V^\star,V}$$

and the set of admissible points

$$X_{\text{ad}} = \{(y,u) \in Y \times U \mid E(y,u) = 0 \;\&\; y(0) = y_\circ \;\&\; \varepsilon u + \mathcal{I}y \le u_b\}.$$

Existence and uniqueness

THEOREM. Assume that the closed, convex and bounded set X_{ad} is nonempty. Then there exists a unique solution $(\bar{y}, \bar{u})$ to (OCP).

Further, there is a unique pair of Lagrange multipliers $(\bar{p}, \bar{\lambda})$ such that

$$\begin{aligned}
\dot{\bar{y}}(t) + \langle \nabla \bar{y}(t), \nabla \cdot \rangle_H - (\mathcal{B}\bar{u} + f)(t) &= 0 \\
\bar{y}(0) - y_\circ &= 0 \\[1ex]
-\dot{\bar{p}}(t) + \langle \nabla \bar{p}(t), \nabla \cdot \rangle_H + (\mathcal{I}^\star \bar{\lambda})(t) + (\bar{y} - y_Q)(t) &= 0 \\
\bar{p}(T) &= 0 \\[1ex]
\kappa \bar{u}(t) - (\mathcal{B}^\star \bar{p})(t) + \varepsilon \bar{\lambda}(t) &= 0 \\[1ex]
\bar{\lambda}(t) - \chi_{\mathcal{A}_b(\bar{y},\bar{p})} \left(\frac{1}{\varepsilon} \mathcal{B}^\star \bar{p} + \frac{\kappa}{\varepsilon} \mathcal{I} \bar{y} - \frac{\kappa}{\varepsilon^2} u_b \right) &= 0
\end{aligned} \tag{OS}$$

where the *active set* $\mathcal{A}_b(\bar{y}, \bar{p})$ is

$$\mathcal{A}_b(y, p) = \left\{ t \in [0, T] \;\middle|\; \frac{\varepsilon}{\kappa} (\mathcal{B}^\star p + \mathcal{I} y)(t) > u_b(t) \right\}.$$

Primal-dual active set strategy (PDASS)

Algorithm (Primal-dual active set strategy)

Require: Initial state-adjoint state pair $(y^0, p^0) \in Y \times Y$.

1: Set $k = 0$
2: **repeat**
3: Calculate active set $\mathcal{A}_b(y^k, p^k)$
4: Solve the primal-dual system (OS) to get (y^{k+1}, p^{k+1})
5: Set $k = k + 1$
6: **until** $\mathcal{A}_b(y^k, p^k) = \mathcal{A}_b(y^{k-1}, p^{k-1})$
7: Return $\bar{u} = u(y^k, p^k)$.

Universität Konstanz

Proper orthogonal decomposition (POD)

Problem: After elimination of u, λ, the discrete linear system (OS) is still of the dimension $2N_tN_x$.

Idea: For $\ell \ll N_x$, find an *optimal* finite element basis $(\psi_1, \dots, \psi_\ell)$ such that the corresponding Galerkin solution y^ℓ is preferably close to $\bar{y}$:

$$\min_{\substack{i=1,\dots,\ell \\ \langle\phi_i,\phi_j\rangle_V=\delta_{ij}}} \Big\| \bar{y} - \underbrace{\sum_{i=1}^{\ell} \langle \bar{y}, \phi_i\rangle_H \phi_i}_{=y^\ell} \Big\|_V^2 . \tag{POD}$$

Realization: Perform an eigenvalue decomposition of a compact, self-adjoint, non-negative operator $\mathcal{K} : V \to V$ which includes the dynamics of the state solution.

Define the canonical mappings $\mathcal{Y} : L^2(0, T; \mathbb{R}) \to V$ and $\mathcal{Y}^\star : V \to L^2(0, T; \mathbb{R})$),

$$\mathcal{Y}\phi = \langle \phi, y\rangle_{L^2(0,T;\mathbb{R})} \in V, \qquad \mathcal{Y}^\star\phi = \langle \phi, y(\cdot)\rangle_V \in L^2(0, T; \mathbb{R})$$

and choose $\mathcal{K} = \mathcal{Y}\mathcal{Y}^\star$.

Proper orthogonal decomposition (POD)

THEOREM. (Continuous version) Let $y \in C(0,T;V)$ an arbitrary state and let $(\lambda_i, \psi_i)_{i\in\mathbb{N}}$ an eigenvalue decomposition of $\mathcal{K} = \mathcal{K}(y)$ with $\lambda_i \geq \lambda_{i+1}$ for all $i \in \mathbb{N}$.
Then $(\psi_i)_{i\in\mathbb{N}}$ is a complete orthonormal system in V, the *rank-ℓ POD basis* $\psi^\ell = (\psi_1, \dots, \psi_\ell)$ is a solution to (POD) and the residual of $y^\ell = y(\psi^\ell)$ sums up to

$$\|y - y^\ell\|^2_{L^2(0,T;V)} = \sum_{i=\ell+1}^{\infty} \lambda_i.$$

Remark. POD can also be provided in H instead of V.

Problem: Good approximation properties of the POD basis are only guaranteed if the POD elements belong to $\mathcal{K}(\bar{y})$. However, $\bar{y}$ is not known, of course. Hence, an adaptive strategy is applied to construct an appropriate POD basis.

Universität Konstanz

Proper orthogonal decomposition (POD)

Discrete POD. Let $Y \in \mathbb{R}^{N_x \times N_t}$ a discrete approximation of y such as

- $y(t_j, x_i) \approx Y_{ij}$ (FDM) or
- $y(t_j, x) \approx \sum_{i=1}^{N_x} Y_{ij}\phi_i(x)$ (FEM).

Further, let $U^\ell \in \mathbb{R}^{N_x \times \ell}$ be the matrix of the first ℓ eigenvectors to

$$\sqrt{W} Y \Theta Y^{\mathrm{T}} \sqrt{W} \in \mathbb{R}^{N_x \times N_x}$$

where $\Theta = \mathrm{diag}(\Delta t) \in \mathbb{R}^{N_t \times N_t}$ provides the discrete $L^2(0, T)$ scalar product and

- $W = \mathrm{diag}(\Delta x) \in \mathbb{R}^{N_x \times N_x}$ (FDM) or
- $W = (\langle \phi_i, \phi_j \rangle_H) \in \mathbb{R}^{N_x \times N_x}$ (FEM)

provides the discrete $L^2(\Omega)$ scalar product. Then a discrete POD basis is given by

$$\psi^\ell = \sqrt{W}^{-1} U^\ell.$$

Remark. If $N_x \gg N_t$ or $\sqrt{W}$ cannot easily be calculated, the eigenvalue decomposition should be provided for the transposed problem

$$\sqrt{\Theta} Y^{\mathrm{T}} W Y \sqrt{\Theta} \in \mathbb{R}^{N_t \times N_t}.$$

Universität Konstanz

Reduced order modeling (ROM)

Algorithm (Reduced order modeling)

Require: $\ell \ll N_x$, initial control guess $u^\ell \in U$, desired exactness ϵ

1: **repeat**
2: Calculate full-order state solution $y = y(u^\ell)$
3: Calculate POD basis $\psi = \psi(y)$
4: Apply PDASS on the reduced system (OS) to get $u^\ell = u(\psi)$
5: **until** Aposti$(u^\ell) < \epsilon$
6: Return $\bar{u} = u^\ell$

Numerical example

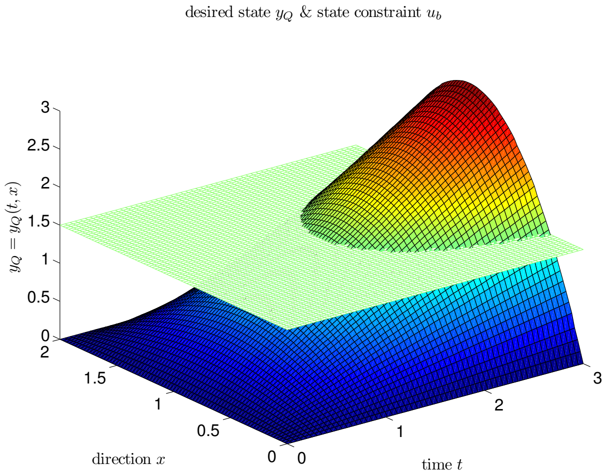


Fig. 1. The desired state $y_Q \in L^2(0,T;H)$ and the upper mixed control-state bound u_b, interpreted as a pure pointwise state constraint.

Numerical example

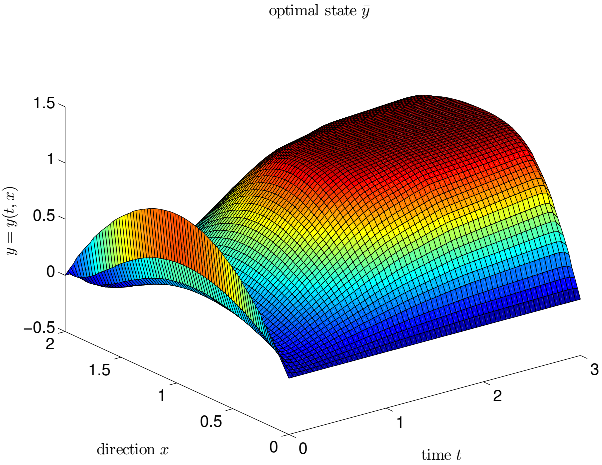


Fig. 2. The optimal state solution $\bar{y} \in Y$ satisfying the state constraints.

Universität Konstanz

Numerical example

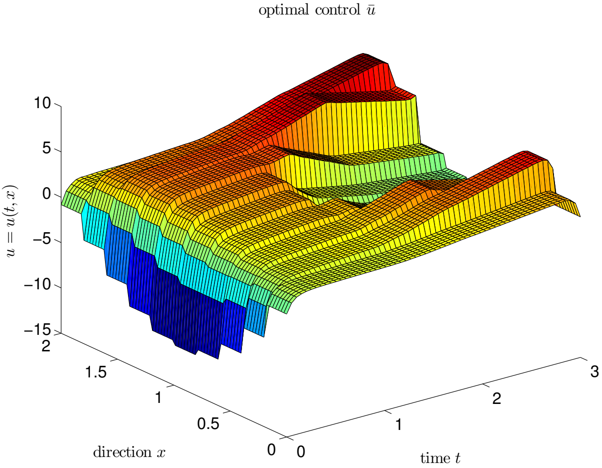


Fig. 3. The optimal control term $\mathcal{B}\bar{u} \in L^2(0, T; H)$ controlling the state equation.

Universität Konstanz

Numerical results

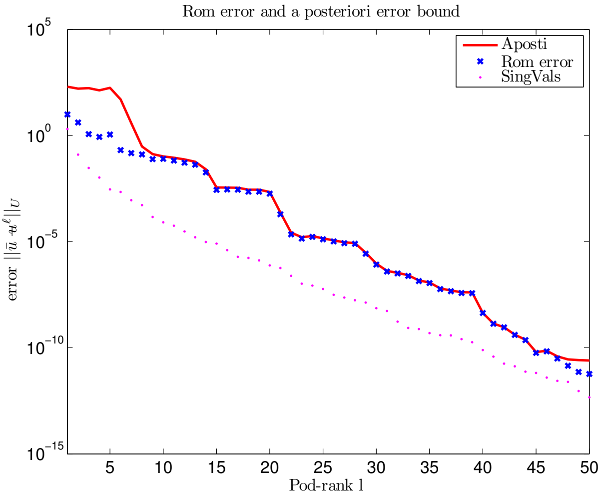
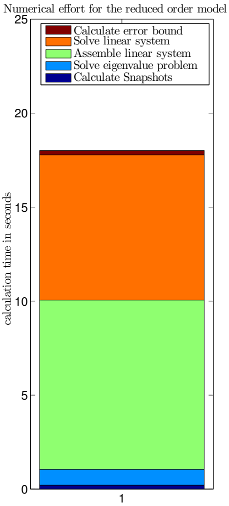


Fig. 4. The ROM errors of the control u^ℓ for different POD basis ranks ℓ, the error bound provided by the a posteriori error estimator and the singular values (squares of the eigenvalues) of the operator $\mathcal{Y}\mathcal{Y}^\star$ which can be used as a costless error indicator.

Universität Konstanz

Numerical effort



Process	Time	#	Total
Assemble full system	0.66 sec	9×	5.97 sec
Solve full system	22.27 sec	9×	200.43 sec
Total			206.40 sec
Solve full snapshots equations	0.11 sec	2×	0.21 sec
Solve eigenvalue problem	0.42 sec	2×	0.84 sec
Assemble ROM system	0.53 sec	17×	9.01 sec
Solve ROM system	0.45 sec	17×	7.72 sec
Evaluate error estimator	0.11 sec	2×	0.23 sec
Total			18.01 sec

Tab. 1. The calculation times for solving the optimization problem with and without model reduction. With 25 POD elements, the ROM problem has to be solved two times; solvings of two eigenvalue problems are required in addition to update the POD basis. Nevertheless, 91.27% of the calculation time is spared in total.

Fig. 5. The numerical effort of the single algorithm processes for solving the ROM problem on an Intel(R) Core(TM) i5 2.40GHz processor.

Universität Konstanz

References

- Hintermüller, M., Ito, K. & Kunisch, K.: *The primal-dual active set strategy as a semismooth Newton method.*
  SIAM J. Optim., **vol. 13**, no. 1: pp. 865–888, 2003.

- Hintermüller, M., Kopacka, I. & Volkwein, S.: *Mesh-independence and preconditioning for solving parabolic control problems with mixed control-state constraints.*
  ESAIM: COVC, **vol. 15**: pp. 626–652, 2009.

- Hinze, M. & Volkwein, S.: *Error estimates for abstract linear-quadratic optimal control problems using proper orthogonal decomposition.*
  Comput. Optim. Appl., **vol. 39**: pp. 319–345, 2008.

- Tröltzsch, F. & Volkwein, S.: *POD a-posteriori error estimates for linear-quadratic optimal control problems.*
  Comput. Optim. Appl., **vol. 44**: pp. 319–345, 2008.

- Volkwein, S.: *Model Reduction using Proper Orthogonal Decomposition.*
  Vorlesungsskript Universität Konstanz, p. 42, 2010.

Universität Konstanz