Optimality System Proper Orthogonal Decomposition for Optimal Control Problems with Control and State Constraints

Seminar on Optimization

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Outline

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Problem formulation

We consider the optimal control problem

$$\min_{y,u,w} J(y,u,w) = \int_\Theta \frac{1}{2}\|y(t) - y_d(t)\|_H^2 + \frac{\sigma_u}{2}\|u(t)\|_{\mathbb{R}^{N_u}}^2 + \frac{\sigma_w}{2}\|w(t)\|_{\mathbb{R}^{N_w}}^2 \,\mathrm{d}t \qquad \text{(OCP)}$$

on the time interval $\Theta = [0,T]$ subject to the linear parabolic pde constraint

$$\langle \dot{y}(t), \varphi\rangle_{V^\star,V} + \langle \mathcal{A}y(t), \varphi\rangle_{V^\star,V} = \langle \mathcal{B}u(t), \varphi\rangle_{V^\star,V} \qquad \forall \varphi \in V$$
$$\langle y(0), \varphi\rangle_H = \langle y_\circ, \varphi\rangle_H \qquad \forall \varphi \in H$$

and the control and state constraints

$$y_a(t) \le \varepsilon w(t) + (\mathcal{I}y)(t) \le y_b(t) \qquad \& \qquad u_a(t) \le u(t) \le u_b(t),$$

with the operators $\mathcal{B} : L^2(\Theta, \mathbb{R}^{N_u}) \to L^2(\Theta, H)$ and $\mathcal{I} : L^2(\Theta, H) \to L^2(\Theta, \mathbb{R}^{N_w})$,

$$(\mathcal{B}u)(t,x) = \sum_{i=1}^{N_u} u_i(t)\chi_i(x), \qquad (\mathcal{I}y)_i(t) = \int_\Omega \pi_i(x)\, y(t,x)\mathrm{d}x.$$

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Transformation on pure box constraints

Introducing a transformed penalty $\omega(t) = \varepsilon w(t) + \mathcal{I}y(t)$, we get the equivalent transformed optimal control problem (TOCP)

$$\min_{y,u,\omega} \tilde{J}(y,u,\omega) = \int_\Theta \frac{1}{2}\|y(t) - \hat{y}_d(t)\|_H^2 + \frac{\sigma_u}{2}\|u(t)\|_{\mathbb{R}^{N_u}}^2 + \frac{\sigma_w}{2\varepsilon^2}\|\omega(t) - \mathcal{I}y(t)\|_{\mathbb{R}^{N_w}}^2 \,\mathrm{d}t$$

subject to the *homogeneous* pde

$$\dot{y}(t) + \mathcal{A}y(t) = \mathcal{B}u(t) \qquad \& \qquad y(0) = 0$$

and the *explicit* penalty and control constraints

$$\hat{y}_a(t) \le \omega(t) \le \hat{y}_b(t) \qquad \& \qquad u_a(t) \le u(t) \le u_b(t)$$

where $\hat{y}_d = y_d - \hat{y}$, $\hat{y}_a = y_a - \mathcal{I}\hat{y}$, $\hat{y}_b = y_b - \mathcal{I}\hat{y}$ and $\hat{y}$ solves

$$\dot{\hat{y}}(t) + \mathcal{A}\hat{y}(t) = 0 \qquad \& \qquad \hat{y}(0) = y_\circ.$$

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Well-posedness and optimality conditions

THEOREM. Assume that the closed, convex and bounded set

$$\{(y,u,\omega) \mid \dot{y} + \mathcal{A}y = \mathcal{B}u \;\&\; y(0) = 0 \;\&\; u \in [u_a, u_b] \;\&\; \omega \in [\hat{y}_a, \hat{y}_b]\}$$

is nonempty. Then there exists a unique solution $(\bar{y}, \bar{u}, \bar{\omega})$ to (TOCP).

Further, defining the *active and inactive sets*

$$\mathcal{A}_a^u = \{t \mid \tfrac{1}{\sigma_u}\mathcal{B}^\star\bar{p} < u_a\}, \qquad \mathcal{A}_b^u = \{t \mid \tfrac{1}{\sigma_u}\mathcal{B}^\star\bar{p} > u_a\}, \qquad \mathcal{A}_i^u = \Theta\backslash(\mathcal{A}_a^u \cup \mathcal{A}_b^u)$$
$$\mathcal{A}_a^y = \{t \mid \mathcal{I}\bar{y} < \hat{y}_a\}, \qquad \mathcal{A}_b^y = \{t \mid \mathcal{I}\bar{y} > \hat{y}_b\}, \qquad \mathcal{A}_i^y = \Theta\backslash(\mathcal{A}_a^y \cup \mathcal{A}_b^y),$$

the following first-order optimality system (OS) is fulfilled:

$$\dot{\bar{y}}(t) + \mathcal{A}y(t) - \mathcal{B}\bar{u}(t) = 0, \qquad \bar{y}(0) = 0,$$
$$-\dot{\bar{p}}(t) + \mathcal{A}p(t) + \tfrac{\sigma_w}{\varepsilon^2}(\mathcal{I}^\star(\mathcal{I}y(t) - \omega(t))) + (y(t) - \hat{y}_d(t)) = 0, \qquad \bar{p}(T) = 0,$$
$$\bar{u}(t) - \tfrac{1}{\sigma_u}\chi_i^u(t)\mathcal{B}^\star\bar{p}(t) - (\chi_a^u(t)u_a(t) + \chi_b^u(t)u_b(t)) = 0,$$
$$\bar{\omega}(t) - \chi_i^y(t)\mathcal{I}\bar{y}(t) - (\chi_a^y(t)\hat{y}_a(t) + \chi_b^y(t)\hat{y}_b(t)) = 0.$$

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Primal-dual active set strategy (PDASS)

Algorithm (Primal-dual active set strategy)

Require: Initial state-adjoint state pair (y^0, p^0).

1: Set $k = 0$
2: **repeat**
3: Calculate the six active and inactive sets with respect to (y^k, p^k).
4: Solve *linear* primal-dual system (OS) with these fix sets to get (y^{k+1}, p^{k+1})
5: Set $k = k + 1$
6: **until** the current and the previous active and inactive sets coincide.
7: Return control $\bar{u} \in L^2(\Theta, \mathbb{R}^{N_u})$ and penalty $\bar{w} = \frac{1}{\varepsilon}(\omega - \mathcal{I}y) \in L^2(\Theta, \mathbb{R}^{N_w})$.

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Proper orthogonal decomposition (POD)

Problem: After elimination of u, ω, the discrete linear system (OS) is still of the dimension $2N_tN_x$.

Idea: For $\ell \ll N_x$, find an *optimal* orthonormal system $\psi = (\psi_1, ..., \psi_\ell) \in V^\ell$ such that the projection error of $\bar{y}$ on the space span(ψ) is minimal:

$$\min_{\phi \in V^\ell \text{ ONB}} \int_\Theta \left\| \bar{y}(t) - \sum_{i=1}^{\ell} \langle \bar{y}(t), \phi_i \rangle_V \phi_i \right\|_V^2 \mathrm{d}t. \qquad \text{(POD)}$$

Realization: Perform an eigenvalue decomposition of a compact, self-adjoint, non-negative operator $\mathcal{R} : V \to V$ which includes the dynamics of the state solution.

Challange: The *optimal* Galerkin ansatz requires the knowledge of the state solution $\bar{y}$ which is not available.

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Proper orthogonal decomposition (POD)

Theorem. (Continuous version) Let $y \in C(0, T; V)$ be an *arbitrary* state and let $(\lambda_i, \psi_i)_{i\in\mathbb{N}}$ be an eigenvalue decomposition of

$$\mathcal{R}(y) : V \to V, \qquad \mathcal{R}(y)\varphi = \int_\Theta \langle y(t), \varphi \rangle_V y(t) \, \mathrm{d}t.$$

with $\lambda_i \geq \lambda_{i+1}$ for all $i \in \mathbb{N}$.

Then $(\psi_i)_{i\in\mathbb{N}}$ is a complete orthonormal system in V and the *rank-ℓ POD basis* $\psi^\ell = (\psi_1, ..., \psi_\ell)$ is a solution to (POD).

A priori estimate: The projection error of y on $V^\ell = \text{span}(\psi)$ fulfills

$$\int_\Theta \left\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \phi_i \rangle_V \phi_i \right\|_V^2 = \sum_{i=\ell+1}^{\infty} \lambda_i.$$

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Proper orthogonal decomposition (POD)

Discrete POD: Let $(t_1, \dots, t_{N_t}) \subseteq \Theta$ be a time discretization scheme with stepsize Δt and let $\varphi = (\varphi_1, \dots, \varphi_{N_x}) \subseteq V$ be a finite element basis with corresponding weights matrix $\mathrm{X} = (\langle \varphi_i, \varphi_j \rangle_V)$. Let $\mathrm{Y} \in \mathbb{R}^{N_x \times N_t}$ be the coefficient matrix of a state

$$y^{\mathrm{FE}}(t_j, x) = \sum_{i=1}^{N_x} \mathrm{Y}_{ij} \varphi_i(x).$$

Then the coefficient matrix of a rank-ℓ POD basis $\psi \in \mathbb{R}^{N_x \times \ell}$ for y^{FE} is given by the discrete eigenvalue problem

$$\Delta t \mathrm{Y} \mathrm{Y}^{\mathrm{T}} \mathrm{X} \psi_l = \lambda_l \psi_l$$

and the l-th POD element in $V^{N_x} = \mathrm{span}(\varphi)$ is represented by

$$\psi_l = \sum_{i=1}^{N_x} \psi_{il} \varphi_i.$$

With $\mathrm{Y}^\ell = \mathrm{Y}^{\mathrm{T}} \mathrm{X} \psi \in \mathbb{R}^{\ell \times N_t}$, the POD approximation y^{POD} of y^{FE} is given by

$$y^{\mathrm{POD}}(t_j, x) = \sum_{l=1}^{\ell} \mathrm{Y}^\ell_{lj} \psi_l(x).$$

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Reduced order model (ROM)

1. **ROM components:**
   - $\mathrm{M} = (\langle \psi_i, \psi_j \rangle_H) \in \mathbb{R}^{\ell \times \ell}$ and $\mathrm{A} = (\langle \mathcal{A} \psi_i, \psi_j \rangle_{V^\star, V}) \in \mathbb{R}^{\ell \times \ell}$.
   - $\mathrm{B} = (\langle \mathcal{B}_j^\star, \psi_i \rangle_{V^\star, V}) \in \mathbb{R}^{\ell \times N_u}$ and $\mathrm{I} = (\mathcal{I}_j \psi_i) \in \mathbb{R}^{N_w \times \ell}$.
   - $\hat{\mathrm{y}}_d(t) = (\langle \hat{y}_d(t), \psi_i \rangle_H) \in \mathbb{R}^\ell$.
2. **ROM system:**

$$\begin{aligned} \mathrm{M}\dot{\mathrm{y}}(t) + \mathrm{A}\mathrm{y}(t) - \mathrm{B}\mathrm{u}(t) &= 0, & \mathrm{y}(0) &= 0, \\ -\mathrm{M}\dot{\mathrm{p}}(t) + \mathrm{A}\mathrm{p}(t) + \tfrac{\sigma_w}{\varepsilon^2}(\mathrm{I}^{\mathrm{T}}(\mathrm{I}\mathrm{y}(t) - \omega(t))) + (\mathrm{M}\mathrm{y}(t) - \hat{\mathrm{y}}_d(t)) &= 0, & \mathrm{p}(T) &= 0, \\ \mathrm{u}(t) - \tfrac{1}{\sigma_u}\chi_i^u(t)\mathrm{B}^{\mathrm{T}}\mathrm{p}(t) - (\chi_i^u(t)u_a(t) + \chi_b^u(t)u_b(t)) &= 0, \\ \omega(t) - \chi_i^y(t)\mathrm{I}\mathrm{y}(t) - (\chi_a^y(t)\hat{y}_a(t) + \chi_b^y(t)\hat{y}_b(t)) &= 0. \end{aligned}$$

3. **ROM expansions:**

$$y^\ell(t) = \sum_{l=1}^{\ell} \mathrm{y}_l(t)\psi_l, \quad p^\ell(t) = \sum_{l=1}^{\ell} \mathrm{p}_l(t)\psi_l, \quad u^\ell = \mathrm{u}, \quad \omega^\ell = \omega$$

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Reduced order model (ROM)

A posteriori error bound: Let (u, ω) be any suboptimal control-penalty pair. Then there exists some computable $\zeta \in L^2(\Theta, \mathbb{R}^{N_u} \times \mathbb{R}^{N_w})$ such that

$$\int_\Theta \|u(t) - \bar{u}(t)\|^2_{\mathbb{R}^{N_u}} + \|\omega(t) - \bar{\omega}(t)\|^2_{\mathbb{R}^{N_w}} \, \mathrm{d}t \leq \frac{1}{\min(\sigma_u^2, \sigma_w^2)} \int_\Theta \|\zeta(t)\|^2_{\mathbb{R}^{N_u} \times \mathbb{R}^{N_w}} \, \mathrm{d}t.$$

In our numerical tests, the evaluation of this error estimator is *cheap* (compared to the effort of solving the optimization problem). Problems may arise here in case of *nonlinear* pdes.

In many applications, the error bounds are *sharp*, i.e. the error bound has the same order as the error itself.

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Reduced order model (ROM)

Algorithm (Model reduction with iterative POD basis updates (IPOD))

Require: Initial control-penalty pair $(u^{(0)}, \omega^{(0)})$, POD basis rank ℓ, desired exactness ε, maximal iteration number $k_{\max}$.

1: Set $k = 0$
2: **repeat**
3: Solve the full state equation for $y^{(k)}$ and the full adjoint state equation for $p^{(k)}$.
4: Solve the POD eigenvalue problem for the rank-ℓ basis $\psi^{(k)}$.
5: Choose PDASS initialization $(\mathrm{y}, \mathrm{p}) = ((\langle y^{(k)}, \psi_l \rangle_H), (\langle p^{(k)}, \psi_l \rangle_H))$ and provide the PDASS algorithm to solve the ROM system; get feedback $(u^{(k)}, \omega^{(k)})$.
6: Set $k = k + 1$
7: **until** $\mathrm{Aposti}(u^{(k)}, \omega^{(k)}) < \varepsilon$ or $k > k_{\max}$.
8: Return control $u^{(k)} \in L^2(\Theta, \mathbb{R}^{N_u})$ and penalty $w^{(k)} = \frac{1}{\varepsilon}(\omega^{(k)} - \mathcal{I}y) \in L^2(\Theta, \mathbb{R}^{N_w})$.

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Optimality system proper orthogonal decomp. (OSPOD)

The optimal state required to determine the POD basis is known *implicitly*:

$$\min_{y,u,\omega,\psi} \tilde{\mathrm{J}}(\mathrm{y},u,\omega,\psi) = \int_\Theta \frac{1}{2}\Big\|\sum_{l=1}^{\ell} \mathrm{y}_l\psi_l - \hat{\mathrm{y}}_d\Big\|_H^2 + \frac{\sigma_u}{2}\|u\|^2_{\mathbb{R}^{N_u}} + \frac{\sigma_w}{2\varepsilon^2}\|\omega - \mathrm{I}(\psi)\mathrm{y}\|^2_{\mathbb{R}^{N_w}}\,\mathrm{d}t$$

subject to the *two* state equations

$$\dot{y} + \mathcal{A}y = \mathcal{B}u, \qquad y(0) = 0, \tag{1}$$

$$\mathrm{M}(\psi)\dot{\mathrm{y}} + \mathrm{A}(\psi)\mathrm{y} = \mathrm{B}(\psi)u, \qquad \mathrm{y}(0) = 0, \tag{2}$$

the POD eigenvalue problem

$$\mathcal{R}(y)\psi_l - \lambda_l\psi_l = 0, \qquad \|\psi_l\|_V^2 = 1 \tag{3}$$

and the penalty and control constraints

$$\hat{y}_a(t) \le \omega(t) \le \hat{y}_b(t) \qquad \& \qquad u_a(t) \le u(t) \le u_b(t).$$

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Optimality system proper orthogonal decomp. (OSPOD)

The corresponding dual system is similar to the optimality equations above:

$$-\dot{p} + \mathcal{A}p + \sum_{l=1}^{\ell}\langle y,\psi_l\rangle_V\mu_l + \sum_{l=1}^{\ell}\langle y,\mu_l\rangle_V\psi_l = 0, \tag{4}$$

$$-\mathrm{M}\dot{\mathrm{p}} + \mathrm{A}\mathrm{p} + \frac{\sigma_w}{\varepsilon^2}\mathrm{I}^{\mathrm{T}}(\mathrm{I}\mathrm{y} - \omega) + (\mathrm{M}\mathrm{y} - \hat{\mathrm{y}}_d) = 0, \tag{5}$$

$$u - \frac{1}{\sigma_u}\chi_i^u(\mathcal{B}^\star p + \mathrm{B}^{\mathrm{T}}\mathrm{p}) - (\chi_a^u u_a + \chi_b^u u_b) = 0, \tag{6}$$

$$\omega - \chi_i^y\mathrm{I}\mathrm{y} - (\chi_a^y\hat{y}_a + \chi_b^y\hat{y}_b) = 0, \tag{7}$$

$$\mathcal{R}(y)\mu_l - \lambda_l\mu_l + \mathcal{N}(\mathrm{y},\mathrm{p},u,\omega,\psi) = 0. \tag{8}$$

with some nonlinear term $\mathcal{N}$ arising by the ψ-differential and the active sets

$$\mathcal{A}_a^u = \{t \mid \tfrac{1}{\sigma_u}(\mathcal{B}^\star\bar{p} + \mathrm{B}^{\mathrm{T}}\mathrm{p}) < u_a\}, \qquad \mathcal{A}_b^u = \{t \mid \tfrac{1}{\sigma_u}(\mathcal{B}^\star\bar{p} + \mathrm{B}^{\mathrm{T}}\mathrm{p}) > u_a\},$$

$$\mathcal{A}_a^y = \{t \mid \mathrm{I}\mathrm{y} < \hat{y}_a\}, \qquad \mathcal{A}_b^y = \{t \mid \mathrm{I}\mathrm{y} > \hat{y}_b\}.$$

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Optimality system proper orthogonal decomp. (OSPOD)

Algorithm (Optimality System POD (OSPOD))

Require: Initial control-penalty pair (u, ω), POD basis rank ℓ, desired exactness ε.
1: **repeat**
2: Solve the full state equation (1) for y.
3: Solve the eigenvalue problem (3) for ψ.
4: Solve the ROM problem (2), (5), (6), (7) for $(\mathrm{y}, \mathrm{p}, u, \omega)$.
5: Solve the "linearized eigenvalue problem" (8) for μ.
6: Solve the full adjoint equation (4) for p.
7: Provide a descent step in direction $-\sigma_u u + \mathcal{B}^\star p + \mathrm{B}^\mathrm{T}\mathrm{p}$ for u
8: Provide a descent step in direction $-\frac{\sigma_w}{\varepsilon^2}(\omega - \mathrm{Iy})$ for ω.
9: **until** $\mathrm{Aposti}(u, \omega) < \varepsilon$.
10: Return control $u \in L^2(\Theta, \mathbb{R}^{N_u})$ and penalty $w = \frac{1}{\varepsilon}(\omega - \mathrm{Iy}) \in L^2(\Theta, \mathbb{R}^{N_w})$.

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Conclusion: POD & OSPOD

POD: Use a *problem specific* Galerkin ansatz with respect to some reference trajectory y.

OSPOD: Use the trajectory of the optimal state $\bar{y}$ to get the *optimal* Galerkin basis.

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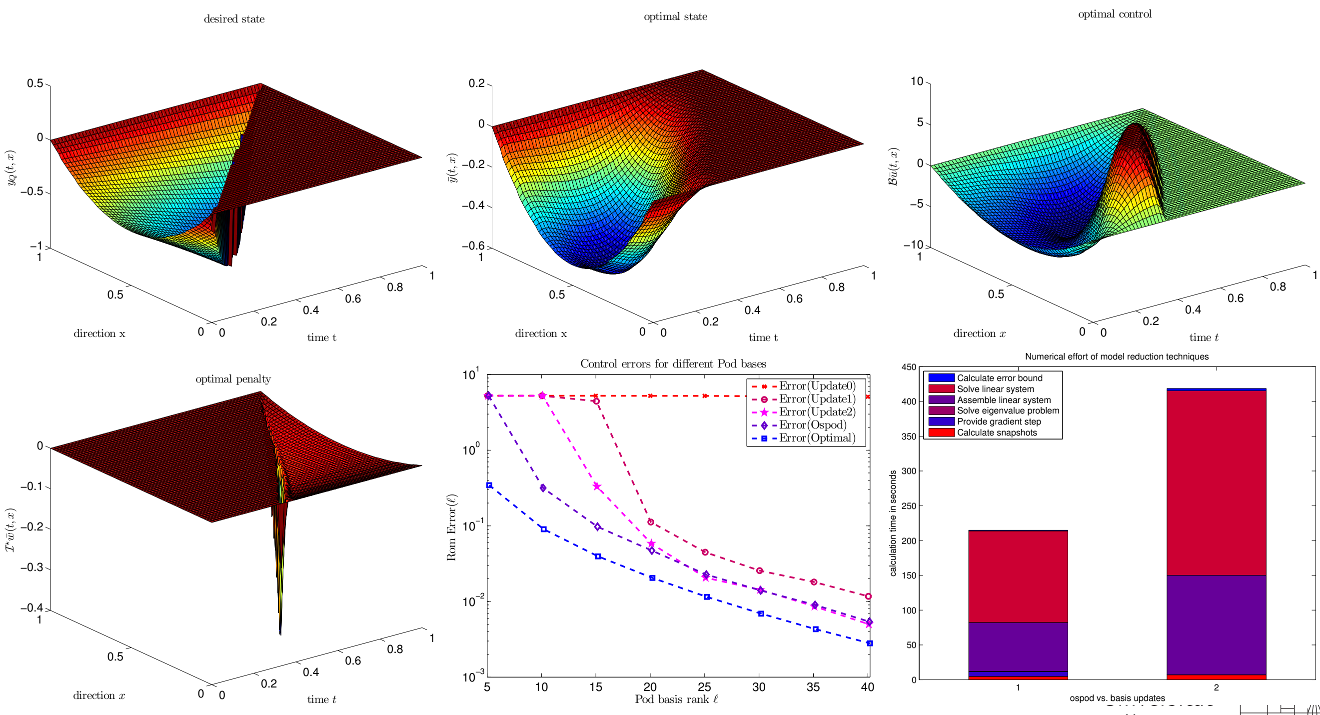
Numerical experiments



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