Model order reduction for Optimal Control Problems

Kolloquium

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Outline

1. The optimal control problem
2. Model reduction
3. Numerical experiments
4. References

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Problem formulation

We consider the optimal control problem (OCP)

$$\min_{y,u,w} J(y,u,w) = \iint\limits_{\Theta\,\Omega} \frac{1}{2}|y(t,x) - y_d(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\sigma_u}{2}\|u\|^2_{L^2(\Theta,\mathbb{R}^\mathsf{m})}$$

subject to the linear parabolic pde constraint

$$\begin{aligned} \dot{y}(t,x) - \Delta y(t,x) &= (\mathcal{B}u)(t,x) && \text{in } \Theta\times\Omega,\\ y(t,x) &= 0 && \text{in } \Theta\times\partial\Omega,\\ y(0,x) &= 0 && \text{in } \Omega \end{aligned}$$

and the control and state constraints

$$y_a \le y(t,x) \le y_b \qquad \& \qquad u_a \le u(t) \le u_b,$$

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and the control and state constraints

$$y_a \le \varepsilon w(t) + (\mathcal{I}y)(t) \le y_b \qquad \& \qquad u_a \le u(t) \le u_b,$$

with the operators $\mathcal{B} : L^2(\Theta,\mathbb{R}^\mathsf{m}) \to L^2(\Theta,H)$ and $\mathcal{I} : L^2(\Theta,H) \to L^2(\Theta,\mathbb{R}^\mathsf{n})$,

$$(\mathcal{B}u)(t,x) = \sum_{i=1}^{\mathsf{m}} u_i(t)\chi_i(x), \qquad (\mathcal{I}y)_i(t) = \int\limits_{\Omega_i} y(t,x)\,\mathrm{d}x.$$

Transformation on pure box constraints

Introducing a transformed penalty $\omega(t) = \varepsilon w(t) + \mathcal{I}y(t)$, we get the equivalent transformed optimal control problem (TOCP)

$$\min_{y,u,\omega} \tilde{J}(y,u,\omega) = \int_\Theta \frac{1}{2}\|y(t) - y_d(t)\|_H^2 \,\mathrm{d}t + \frac{\sigma_u}{2}\|u(t)\|_{\mathbb{R}^m}^2 + \frac{\sigma_w}{2\varepsilon^2}\|\omega(t) - \mathcal{I}y(t)\|_{\mathbb{R}^n}^2$$

subject to the evolution equation

$$E(y,u)(t) = \dot{y}(t) - \Delta y(t) - \mathcal{B}u(t) = 0 \text{ in } V' \qquad \& \qquad y(0) = 0 \text{ in } H$$

with $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and the *explicit* penalty and control constraints

$$\hat{y}_a \le \omega(t) \le \hat{y}_b \qquad \& \qquad u_a \le u(t) \le u_b.$$

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Well-posedness and optimality conditions

THEOREM. There exists a unique solution $(\bar{y}, \bar{u}, \bar{\omega})$ to (TOCP).

Introducing the Langange function

$$\mathscr{L}(y,u,\omega,p) = \tilde{J}(y,u,\omega) + \int_\Theta \langle E(y,u)(t), p(t)\rangle_{V',V} \,\mathrm{d}t,$$

we get the variational optimality conditions

$$\begin{aligned}
\mathscr{L}_p(\bar{y},\bar{u},\bar{\omega},\bar{p}) &= 0; \\
\mathscr{L}_y(\bar{y},\bar{u},\bar{\omega},\bar{p}) &= 0; \\
\mathscr{L}_u(\bar{y},\bar{u},\bar{\omega},\bar{p})(u - \bar{u}) &\ge 0 \qquad \forall u \in [u_a, u_b]; \\
\mathscr{L}_\omega(\bar{y},\bar{u},\bar{\omega},\bar{p})(\omega - \bar{\omega}) &\ge 0 \qquad \forall \bar{\omega} \in [\hat{y}_a, \hat{y}_b].
\end{aligned}$$

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Well-posedness and optimality conditions

With linear operators $\mathcal{L}_1, \mathcal{L}_2$ and nonlinear operators $\mathcal{N}_1, \mathcal{N}_2$, the following first-order optimality conditions are satisfied:

$$\begin{aligned} \dot{y} - \Delta y - \mathcal{B}u &= 0 \\ -\dot{p} - \Delta p - \mathcal{L}_1 y - \mathcal{L}_2 \omega &= 0 \\ u - \mathcal{N}_1(p)p &= 0 \\ \omega - \mathcal{N}_2(y)y &= 0 \end{aligned}$$

The system can be iteratively solved by the primal-dual active set strategy (PDASS)

$$\begin{aligned} \dot{y}_{k+1} - \Delta y_{k+1} &= \mathcal{B}\mathcal{N}_1(p_k)p_{k+1} \\ -\dot{p}_{k+1} - \Delta p_{k+1} &= (\mathcal{L}_1 + \mathcal{L}_2\mathcal{N}_2(y_k))y_{k+1}. \end{aligned}$$

This is a semismooth Newton method with global convergence and superlinear convergence rates.

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Proper orthogonal decomposition (POD)

Variational form: For all test functions $\varphi \in V$ we postulate

$$\begin{aligned} \langle \dot{y} - \Delta y - \mathcal{B}\mathcal{N}_1 p, \varphi \rangle_{V',V} &= 0, \\ \langle -\dot{p} - \Delta p - (\mathcal{L}_1 + \mathcal{L}_2\mathcal{N}_2)y, \varphi \rangle_{V',V} &= 0. \end{aligned}$$

Discretization: Replace the test space V by a finite dimensional subspace $V^N \subseteq V$ spanned by $\varphi_1, ..., \varphi_N \in V$ and search approximate solutions y^N, p^N of the form

$$y^N(t,x) = \sum_{i=1}^N \mathrm{y}_i(t)\varphi_i \qquad \& \qquad p^N(t,x) = \sum_{i=1}^N \mathrm{p}_i(t)\varphi_i.$$

Problem: The discretized linear systems are still of the dimension $2N$.

Idea: For $\ell \ll N$, find an *optimal* orthonormal system $\psi = (\psi_1, ..., \psi_\ell) \subseteq V$ such that the projection error of y on the space $\mathrm{span}(\psi)$ is minimal:

$$\min_{\psi \text{ ONB}} \int_\Theta \left\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \psi_i \rangle_V \psi_i \right\|_V^2 \mathrm{d}t. \tag{POD}$$

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Proper orthogonal decomposition (POD)

THEOREM. (Continuous version) Let $y \in C(0, T; V)$ be an *arbitrary* state and let $(\lambda_i, \psi_i)_{i \in \mathbb{N}}$ be a normalized eigenvalue decomposition of the compact, nonnegative, selfadjoint operator

$$\mathcal{R}(y) : V \to V, \qquad \mathcal{R}(y)\varphi = \int_\Theta \langle y(t), \varphi \rangle_V y(t)\, \mathrm{d}t.$$

with $\lambda_i \geq \lambda_{i+1}$ for all $i \in \mathbb{N}$.

Then the *rank-ℓ POD basis* $\psi^\ell = (\psi_1, ..., \psi_\ell)$ is a solution to (POD).

A priori estimate: The projection error of y on $V^\ell = \operatorname{span}(\psi)$ fulfills

$$\int_\Theta \left\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \phi_i \rangle_V \phi_i \right\|_V^2 = \sum_{i=\ell+1}^{\infty} \lambda_i.$$

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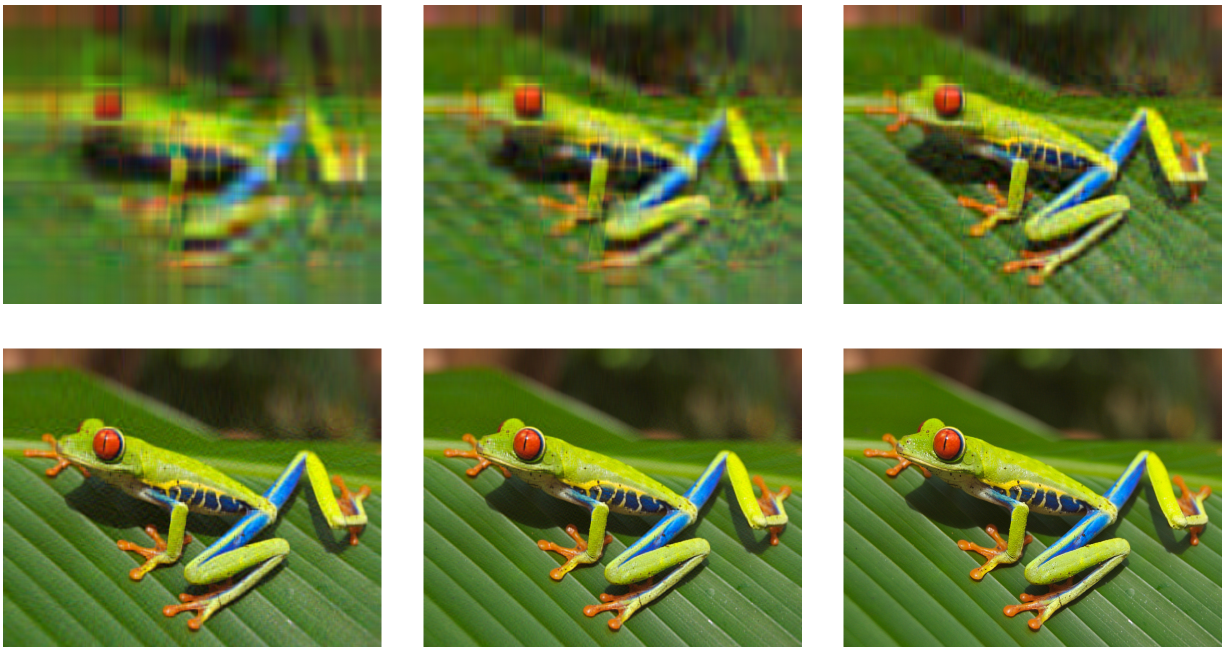
Filtering and compression of frogs



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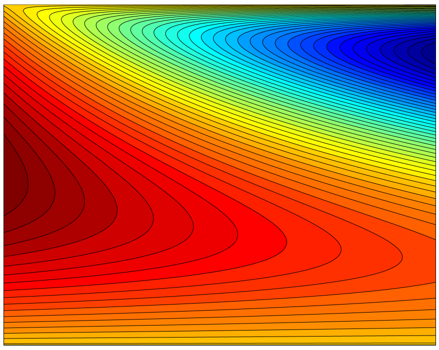
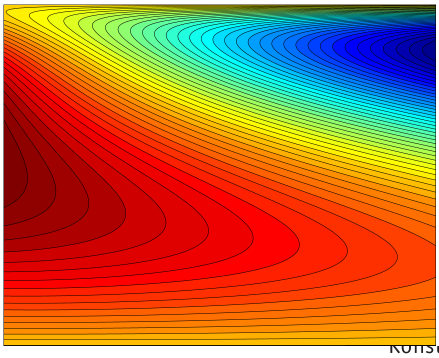
Random data and dynamical flow



Reduced order model (ROM)

1. **ROM components:**
   - $\mathrm{M} = (\langle \psi_i, \psi_j \rangle_H) \in \mathbb{R}^{\ell \times \ell}$ and $\mathrm{A} = (\langle -\Delta \psi_i, \psi_j \rangle_{V^\star, V}) \in \mathbb{R}^{\ell \times \ell}$.
   - $\mathrm{L}_1 \in \mathrm{R}^{\ell \times \ell}$, $\mathrm{L}_2 \in \mathrm{R}^{\ell \times \mathrm{n}}$, $\mathrm{N}_1(\mathrm{p}) \in \mathrm{R}^{\mathrm{m} \times \ell}$, $\mathrm{N}_2(\mathrm{y}) \in \mathrm{R}^{\mathrm{n} \times \ell}$ and $\mathrm{B} = (\langle \mathcal{B}_j^\star, \psi_i \rangle_{V^\star, V}) \in \mathbb{R}^{\ell \times \mathrm{m}}$.

2. **ROM system:**

$$\begin{aligned} \mathrm{M}\dot{\mathrm{y}}(t) + \mathrm{A}\mathrm{y}(t) - \mathrm{B}\mathrm{N}_1(\mathrm{p})\mathrm{p}(t) &= 0, & \mathrm{y}(0) &= 0, \\ -\mathrm{M}\dot{\mathrm{p}}(t) + \mathrm{A}\mathrm{p}(t) - (\mathrm{L}_1 + \mathrm{L}_2\mathrm{N}_2(\mathrm{y}))\mathrm{y} &= 0, & \mathrm{p}(T) &= 0. \end{aligned}$$

3. **ROM expansions:**

$$y^\ell(t) = \sum_{l=1}^{\ell} \mathrm{y}_l(t)\psi_l, \quad p^\ell(t) = \sum_{l=1}^{\ell} \mathrm{p}_l(t)\psi_l,$$

$$u^\ell(t) = \mathcal{N}_1(p^\ell)p^\ell, \quad \omega^\ell(t) = \mathcal{N}_2(y^\ell)y^\ell.$$

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Reduced order model (ROM)

A posteriori error bound: Let (u, ω) be any suboptimal control-penalty pair. Then there exists some computable $\zeta \in L^2(\Theta, \mathbb{R}^m \times \mathbb{R}^n)$ such that

$$\int_\Theta \|u(t) - \bar{u}(t)\|^2_{\mathbb{R}^m} + \|\omega(t) - \bar{\omega}(t)\|^2_{\mathbb{R}^n} \,\mathrm{d}t$$
$$\leq \frac{1}{\min(\sigma_u^2, \sigma_w^2)} \int_\Theta \|\zeta(t)\|^2_{\mathbb{R}^m \times \mathbb{R}^n} \,\mathrm{d}t + C(\Delta t + \Delta x^2).$$

Similar results are available for nonlinear PDEs; then second-order information – the smallest eigenvalue of the Hessian $\tilde{J}''$ – is required.

$\rightarrow$ SQP (Sequential Quadratic Programming) & TR-POD (Trust Region POD).

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Reduced order model (ROM)

Algorithm (Model reduction with iterative POD basis updates (IPOD))

Require: Initial control-penalty pair $(u^{(0)}, \omega^{(0)})$, POD basis rank ℓ, desired exactness ε, maximal iteration number $k_{\max}$.

Set $k = 0$
repeat
  Solve the full state equation for $y^{(k)}$.
  Solve the POD eigenvalue problem for the rank-ℓ basis $\psi^{(k)}$.
  Provide the PDASS algorithm to solve the ROM system for $(u^{(k)}, \omega^{(k)})$.
  Set $k = k + 1$
until Aposti$(u^{(k)}, \omega^{(k)}) < \varepsilon$ or $k > k_{\max}$.
Return control $u^{(k)} \in L^2(\Theta, \mathbb{R}^m)$ and penalty $w^{(k)} = \frac{1}{\varepsilon}(\omega^{(k)} - \mathcal{I}y) \in L^2(\Theta, \mathbb{R}^n)$.

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Optimality system proper orthogonal decomp. (OSPOD)

The optimal state required to determine the POD basis is known *implicitly*:

$$\min_{y,u,\omega,\psi} \tilde{\mathrm{J}}(\mathrm{y},u,\omega,\psi) = \int_\Theta \frac{1}{2}\Big\|\sum_{l=1}^{\ell} \mathrm{y}_l\psi_l - y_d\Big\|_H^2 + \frac{\sigma_u}{2}\|u\|_{\mathbb{R}^m}^2 + \frac{\sigma_w}{2\varepsilon^2}\|\omega - \mathrm{I}(\psi)\mathrm{y}\|_{\mathbb{R}^n}^2 \,\mathrm{d}t$$

subject to the *two* state equations

$$\dot{y} + \mathcal{A}y = \mathcal{B}u, \qquad y(0) = 0, \tag{1}$$

$$\mathrm{M}(\psi)\dot{\mathrm{y}} + \mathrm{A}(\psi)\mathrm{y} = \mathrm{B}(\psi)u, \qquad \mathrm{y}(0) = 0, \tag{2}$$

the POD eigenvalue problem

$$\mathcal{R}(y)\psi_l - \lambda_l\psi_l = 0, \qquad \|\psi_l\|_V^2 = 1 \tag{3}$$

and the penalty and control constraints

$$y_a(t) \le \omega(t) \le y_b(t) \qquad \& \qquad u_a(t) \le u(t) \le u_b(t).$$

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A priori error estimates

If POD is provided with the snapshots $y(\bar{u})$, we have the a priori error estimate

$$\|y(u) - y^\ell(u)\|_{L^2(\Theta,V)}^2 \le C(\bar{u})\Big(\sum_{l=\ell+1}^{\infty}\lambda_l + \|y_\circ - \mathcal{P}^\ell y_\circ\|_H^2 + \int_\Theta \|\dot{y}(u) - \mathcal{P}^\ell\dot{y}(u)\|_V^2\,\mathrm{d}t\Big)$$

where $\mathcal{P}^\ell : V \to V^\ell$ is the projection $\phi \mapsto \sum_{l=1}^{\ell}\langle\phi,\psi_l\rangle_V\psi_l$.

With $y_\circ = 0$ and the usage of the additional snapshots $\dot{y}(\bar{u})$, OS-POD admits decay rates of the form

$$\|y(u) - y^\ell(u)\|_{L^2(\Theta,V)}^2 \le \frac{C}{\ell}\|u\|_U^2.$$

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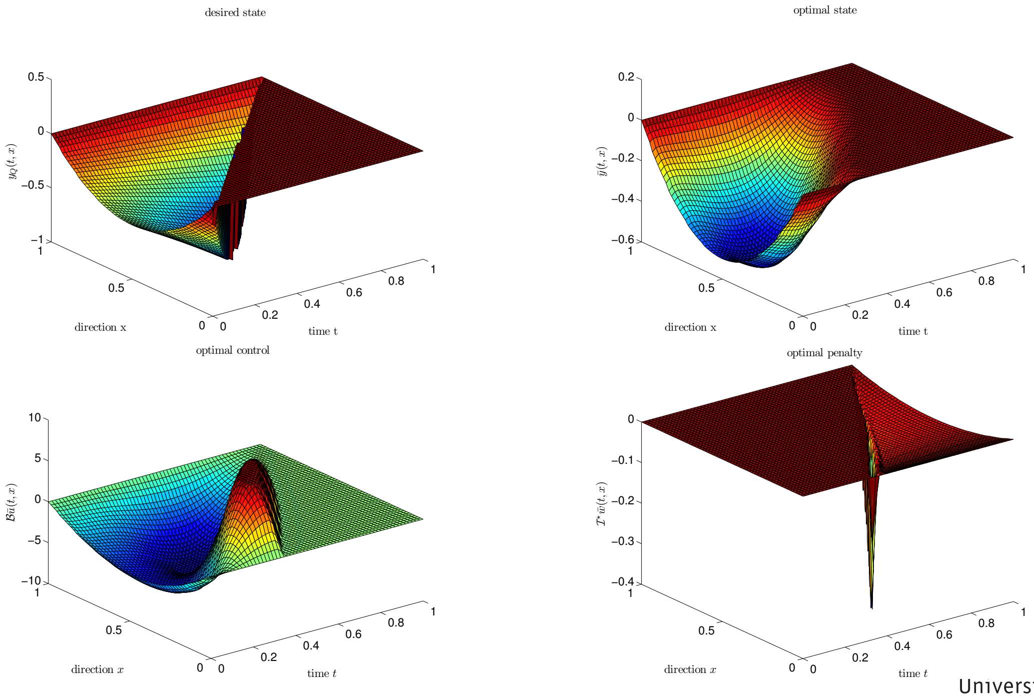
Numerical experiments: Run 1



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Calculation times

method	DoF	CPU time	relative time
finite element system	$N_x = 500$	860.75 sec	100.00%
initial basis	$\ell = 35$	110.77 sec	13.02%
iterative basis updates	$\ell = 15$	37.41 sec	4.40%
OS-POD basis selection	$\ell = 13$	18.39 sec	2.16%
optimal POD basis	$\ell = 13$	11.48 sec	1.35%

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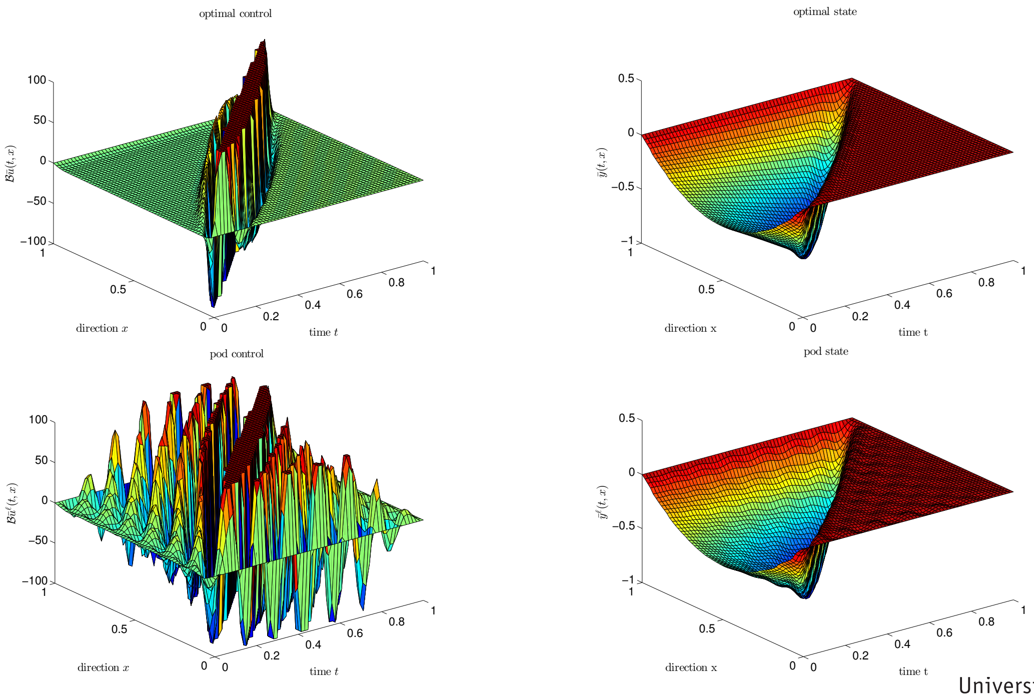
Numerical experiments: Run 2



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