POD Model Order Reduction for Optimal Control Problems

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Outline

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Problem formulation

We consider the optimal control problem (OCP)

$$\min_{y,u,w} J(y,u,w) = \iint_{\Theta\,\Omega} \frac{1}{2}|y(t,x) - y_d(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\sigma_u}{2}\|u\|^2_{L^2(\Theta,\mathbb{R}^{\mathsf{m}})} + \frac{\sigma_w}{2}\|w(t)\|^2_{L^2(\Theta,\mathbb{R}^{\mathsf{n}})}$$

subject to the linear parabolic pde constraint

$$\begin{aligned} \dot{y}(t,x) - \Delta y(t,x) &= (\mathcal{B}u)(t,x) && \text{in } \Theta\times\Omega, \\ y(t,x) &= 0 && \text{in } \Theta\times\partial\Omega, \\ y(0,x) &= 0 && \text{in } \Omega \end{aligned}$$

and the control and state constraints

$$y_a \le \varepsilon w(t) + (\mathcal{I}y)(t) \le y_b \qquad \& \qquad u_a \le u(t) \le u_b,$$

with the operators $\mathcal{B} : L^2(\Theta,\mathbb{R}^{\mathsf{m}}) \to L^2(\Theta,H)$ and $\mathcal{I} : L^2(\Theta,H) \to L^2(\Theta,\mathbb{R}^{\mathsf{n}})$,

$$(\mathcal{B}u)(t,x) = \sum_{i=1}^{\mathsf{m}} u_i(t)\chi_i(x), \qquad (\mathcal{I}y)_i(t) = \int_{\Omega_i} y(t,x)\,\mathrm{d}x.$$

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Well-posedness and optimality conditions

THEOREM. There exists a unique solution $(\bar{y}, \bar{u}, \bar{w})$ to (OCP).

THEOREM. With the transformation $\omega = \varepsilon w + \mathcal{I}y$, linear operators $\mathcal{L}_1, \mathcal{L}_2$ and nonlinear operators $\mathcal{N}_1, \mathcal{N}_2$, (OCP) admits regular Lagrange multipliers and the following first-order optimality conditions are satisfied:

$$
\begin{aligned}
\dot{y} - \Delta y - \mathcal{L}_1(u) &= 0, & u - \mathcal{N}_1(p)p &= 0 \\
-\dot{p} - \Delta p - \mathcal{L}_2(y, \omega) &= 0, & \omega - \mathcal{N}_2(y)y &= 0
\end{aligned}
$$

The system can be solved iteratively by the primal-dual active set strategy (PDASS)

$$
\begin{aligned}
\dot{y}_{k+1} - \Delta y_{k+1} &= \mathcal{L}_1 \mathcal{N}_1(p_k) p_{k+1} \\
-\dot{p}_{k+1} - \Delta p_{k+1} &= \mathcal{L}_2(y_{k+1}, \mathcal{N}_2(y_k) y_{k+1}).
\end{aligned}
$$

This is a semismooth Newton method with global convergence and superlinear convergence rates.

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Proper orthogonal decomposition (POD)

Discretization: Let $V^\ell \subseteq V$ be an ℓ-dimensional subspace of V. For all test functions $\varphi \in V^\ell$ we consider the variational equation

$$\langle \dot{y} - \Delta y - \mathcal{B}u, \varphi \rangle_{V',V} = 0.$$

We look for an optimal orthonormal system $\psi = (\psi_1, ..., \psi_\ell) \subseteq V$ such that the projection error of y on the space $V^\ell = \mathrm{span}(\psi)$ is minimal:

$$\min_{\psi \text{ ONB}} \int_\Theta \left\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \psi_i \rangle_V \psi_i \right\|_V^2 \mathrm{d}t. \qquad \text{(POD)}$$

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Proper orthogonal decomposition (POD)

THEOREM. Let $(\lambda_i, \psi_i)_{i\in\mathbb{N}}$ be a normalized eigenvalue decomposition of the compact, nonnegative, selfadjoint operator

$$\mathcal{R}(y) : V \to V, \qquad \mathcal{R}(y)\varphi = \int_\Theta \langle y(t), \varphi\rangle_V y(t)\,\mathrm{d}t.$$

with $\lambda_i \geq \lambda_{i+1}$ for all $i \in \mathbb{N}$.

Then the rank-ℓ POD basis $\psi^\ell = (\psi_1, ..., \psi_\ell)$ is a solution to (POD).

A-priori estimate: The projection error of y on $V^\ell = \mathrm{span}(\psi)$ fulfills

$$\int_\Theta \left\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \phi_i\rangle_V \phi_i \right\|_V^2 = \sum_{i=\ell+1}^{\infty} \lambda_i.$$

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Reduced order model (ROM)

Let (u^ℓ, ω^ℓ) be the solution to the reduced system in V^ℓ.

A-posteriori error bound: There exists some computable $\zeta \in L^2(\Theta, \mathbb{R}^m \times \mathbb{R}^n)$ with

$$\int_\Theta \|u(t) - \bar{u}(t)\|^2_{\mathbb{R}^m} + \|\omega(t) - \bar{\omega}(t)\|^2_{\mathbb{R}^n} \, \mathrm{d}t \leq \int_\Theta \|\zeta(t)\|^2_{\mathbb{R}^m \times \mathbb{R}^n} \, \mathrm{d}t + C(\Delta t + \Delta x^2).$$

Further, $(u^\ell, \omega^\ell) \to (\bar{u}, \bar{\omega})$ for $\ell \to \infty$ and ζ vanishes with the same rate.

Similar results are available for nonlinear PDEs; then second-order information – the smallest eigenvalue of the Hessian J'' – is required.

$\rightarrow$ SQP (Sequential Quadratic Programming) & TR-POD (Trust Region POD).

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Optimality system proper orthogonal decomp. (OSPOD)

The optimal state required to determine the POD basis is known implicitly:

$$\min_{y,u,\omega,\psi} \mathrm{J}(\mathrm{y},u,\omega,\psi) = \int_\Theta \frac{1}{2}\Big\| \sum_{l=1}^{\ell} \mathrm{y}_l\psi_l - y_d \Big\|_H^2 + \frac{\sigma_u}{2}\|u\|_{\mathbb{R}^m}^2 + \frac{\sigma_w}{2\varepsilon^2}\|\omega - \mathrm{I}(\psi)\mathrm{y}\|_{\mathbb{R}^n}^2 \,\mathrm{d}t$$

subject to the full-order state equation

$$\dot{y} + \mathcal{A}y = \mathcal{B}u \qquad y(0) = 0,$$

the reduced-order state equation

$$\mathrm{M}(\psi)\dot{\mathrm{y}} + \mathrm{A}(\psi)\mathrm{y} = \mathrm{B}(\psi)u, \qquad \mathrm{y}(0) = 0,$$

the POD eigenvalue problem

$$\mathcal{R}(y)\psi_l - \lambda_l\psi_l = 0, \qquad \|\psi_l\|_V^2 = 1$$

and the penalty and control constraints

$$y_a(t) \le \omega(t) \le y_b(t) \qquad \& \qquad u_a(t) \le u(t) \le u_b(t).$$

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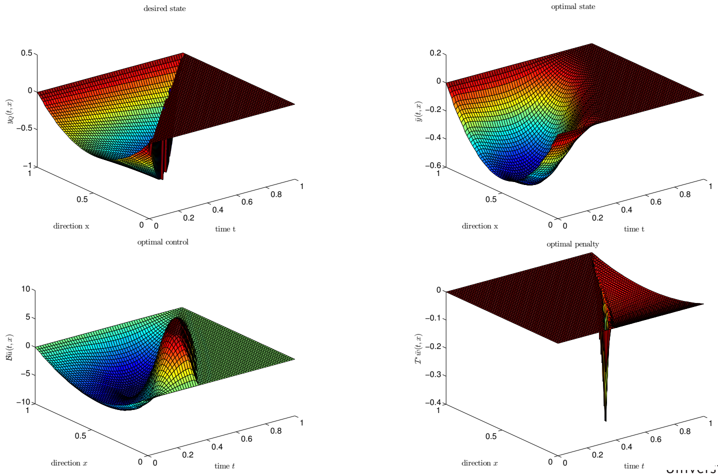
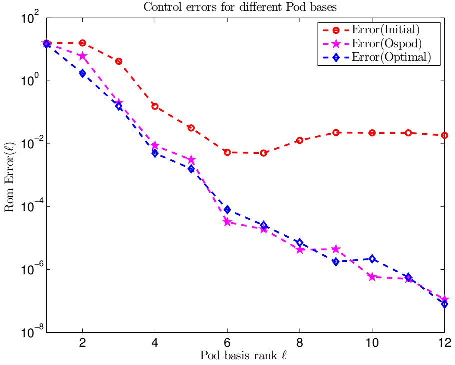
Optimality system proper orthogonal decomp. (OSPOD)

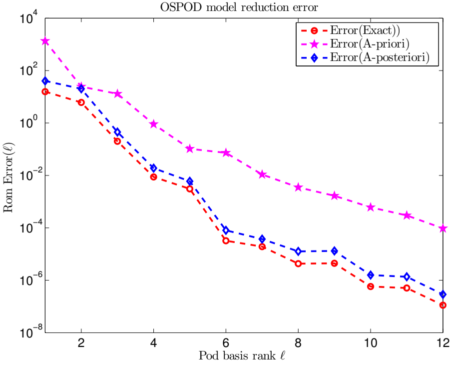
1. The OSPOD system is solved iteratively; the semismooth Newton method is applied to the coupled reduced components where a gradient method is provided for the uncoupled full-order part.
2. Since the OSPOD basis belongs to the optimal state, a-priori bounds are applicable in addition to the a-posteriori analysis.
3. To guarantee convergence of the iterative ansatz, perturbation arguments are used.

Numerical experiments



Numerical experiments





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Numerical experiments

method	DoF	CPU time	relative time
finite element system	$N_x = 500$	860.75 sec	100.00%
initial basis	$\ell = 35$	110.77 sec	13.02%
iterative basis updates	$\ell = 15$	37.41 sec	4.40%
OS-POD basis selection	$\ell = 13$	18.39 sec	2.16%
optimal POD basis	$\ell = 13$	11.48 sec	1.35%

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