POD Model Order Reduction for Optimal Control Problems

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Outline

1. The optimal control problem
2. Model reduction
3. Numerical experiments
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Problem formulation

We consider the optimal control problem (OCP)

$$
\min_{y,u,w} J(y, u, w) = \int_{\Theta} \int_{\Omega} \frac{1}{2} |y(t, x) - y_d(t, x)|^2 \, dx \, dt + \frac{\sigma_u}{2} \|u\|_{L^2(\Theta, \mathbb{R}^m)}^2 + \frac{\sigma_w}{2} \|w(t)\|_{L^2(\Theta, \mathbb{R}^n)}^2
$$

subject to the linear parabolic pde constraint

$$
\dot{y}(t, x) - \Delta y(t, x) = (Bu)(t, x) \quad \text{in } \Theta \times \Omega,
$$
$$
y(t, x) = 0 \quad \text{in } \Theta \times \partial \Omega,
$$
$$
y(0, x) = 0 \quad \text{in } \Omega
$$

and the control and state constraints

$$
y_a \leq \varepsilon w(t) + (Iy)(t) \leq y_b \quad \& \quad u_a \leq u(t) \leq u_b,
$$

with the operators $B : L^2(\Theta, \mathbb{R}^m) \rightarrow L^2(\Theta, H)$ and $I : L^2(\Theta, H) \rightarrow L^2(\Theta, \mathbb{R}^n)$,

$$
(Bu)(t, x) = \sum_{i=1}^{m} u_i(t) \chi_i(x), \quad (Iy)_i(t) = \int_{\Omega_i} y(t, x) \, dx.
$$
Well-posedness and optimality conditions

Theorem. There exists a unique solution \((\bar{y}, \bar{u}, \bar{w})\) to (OCP).

Theorem. With the transformation \(\omega = \varepsilon w + \mathcal{L}y\), linear operators \(\mathcal{L}_1, \mathcal{L}_2\) and nonlinear operators \(\mathcal{N}_1, \mathcal{N}_2\), (OCP) admits regular Lagrange multipliers and the following first-order optimality conditions are satisfied:

\[
\begin{align*}
\dot{y} - \Delta y - \mathcal{L}_1(u) &= 0, \\
-\dot{p} - \Delta p - \mathcal{L}_2(y, \omega) &= 0
\end{align*}
\]

\[
\begin{align*}
u - \mathcal{N}_1(p)p &= 0, \\
\omega - \mathcal{N}_2(y)y &= 0
\end{align*}
\]

The system can be solved iteratively by the primal-dual active set strategy (PDASS)

\[
\begin{align*}
\dot{y}_{k+1} - \Delta y_{k+1} &= \mathcal{L}_1\mathcal{N}_1(p_k)p_{k+1} \\
-\dot{p}_{k+1} - \Delta p_{k+1} &= \mathcal{L}_2(y_{k+1}, \mathcal{N}_2(y_k)y_{k+1})
\end{align*}
\]

This is a semismooth Newton method with global convergence and superlinear convergence rates.
Proper orthogonal decomposition (POD)

**Discretization:** Let $V^\ell \subseteq V$ be an $\ell$-dimensional subspace of $V$. For all test functions $\varphi \in V^\ell$ we consider the variational equation

$$\langle \dot{y} - \Delta y - Bu, \varphi \rangle_{V', V} = 0.$$ 

We look for an optimal orthonormal system $\psi = (\psi_1, ..., \psi_\ell) \subseteq V$ such that the projection error of $y$ on the space $V^\ell = \text{span}(\psi)$ is minimal:

$$\min_{\psi \text{ ONB}} \int_\Theta \left\| y(t) - \sum_{i=1}^\ell \langle y(t), \psi_i \rangle_{V} \psi_i \right\|_V^2 \, dt.$$  

(POD)
Proper orthogonal decomposition (POD)

**Theorem.** Let \((\lambda_i, \psi_i)_{i \in \mathbb{N}}\) be a normalized eigenvalue decomposition of the compact, nonnegative, selfadjoint operator

\[
\mathcal{R}(y) : V \rightarrow V, \quad \mathcal{R}(y) \varphi = \int_{\Theta} \langle y(t), \varphi \rangle dy(t) dt.
\]

with \(\lambda_i \geq \lambda_{i+1}\) for all \(i \in \mathbb{N}\).

Then the rank-\(\ell\) POD basis \(\psi^\ell = (\psi_1, ..., \psi_\ell)\) is a solution to (POD).

**A-priori estimate:** The projection error of \(y\) on \(V^\ell = \text{span}(\psi)\) fulfills

\[
\int_{\Theta} \left\| y(t) - \sum_{i=1}^\ell \langle y(t), \phi_i \rangle V \phi_i \right\|^2_V = \sum_{i=\ell+1}^{\infty} \lambda_i.
\]
Reduced order model (ROM)

Let \((u^\ell, \omega^\ell)\) be the solution to the reduced system in \(V^\ell\).

**A-posteriori error bound:** There exists some computable \(\zeta \in L^2(\Theta, \mathbb{R}^m \times \mathbb{R}^n)\) with

\[
\int_{\Theta} \|u(t) - \bar{u}(t)\|_{\mathbb{R}^m}^2 + \|\omega(t) - \bar{\omega}(t)\|_{\mathbb{R}^n}^2 \, dt \leq \int_{\Theta} \|\zeta(t)\|_{\mathbb{R}^m \times \mathbb{R}^n}^2 \, dt + C(\Delta t + \Delta x^2).
\]

Further, \((u^\ell, \omega^\ell) \to (\bar{u}, \bar{\omega})\) for \(\ell \to \infty\) and \(\zeta\) vanishes with the same rate.

Similar results are available for nonlinear PDEs; then second-order information – the smallest eigenvalue of the Hessian \(J''\) – is required.

\(\rightarrow\) SQP (Sequential Quadratic Programming) & TR-POD (Trust Region POD).
Optimality system proper orthogonal decomp. (OSPOD)

The optimal state required to determine the POD basis is known implicitly:

\[
\min_{y,u,\omega,\psi} J(y, u, \omega, \psi) = \int_{\Theta} \left\{ \frac{1}{2} \left| \sum_{l=1}^{\ell} y_l \psi_l - y_d \right|_H^2 + \frac{\sigma_u}{2} \| u \|_{\mathbb{R}^m}^2 + \frac{\sigma_w}{2\varepsilon^2} \| \omega - I(\psi)y \|_{\mathbb{R}^n}^2 \right\} dt
\]

subject to the full-order state equation

\[
\dot{y} + Ay = Bu \quad y(0) = 0,
\]

the reduced-order state equation

\[
M(\psi)\dot{y} + A(\psi)y = B(\psi)u, \quad y(0) = 0,
\]

the POD eigenvalue problem

\[
\mathcal{R}(y)\psi_l - \lambda_l \psi_l = 0, \quad \|\psi_l\|_V^2 = 1
\]

and the penalty and control constraints

\[
y_a(t) \leq \omega(t) \leq y_b(t) \quad \& \quad u_a(t) \leq u(t) \leq u_b(t).
\]
The OSPOD system is solved iteratively; the semismooth Newton method is applied to the coupled reduced components where a gradient method is provided for the uncoupled full-order part.

Since the OSPOD basis belongs to the optimal state, a-priori bounds are applicable in addition to the a-posteriori analysis.

To guarantee convergence of the iterative ansatz, perturbation arguments are used.
Numerical experiments
### Numerical experiments

#### Control errors for different Pod bases

<table>
<thead>
<tr>
<th>Pod basis rank $\ell$</th>
<th>Rom Error($\ell$)</th>
</tr>
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<tbody>
<tr>
<td>$2$</td>
<td>$10^2$</td>
</tr>
<tr>
<td>$4$</td>
<td>$10^0$</td>
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<tr>
<td>$6$</td>
<td>$10^{-2}$</td>
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<td>$10$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>$12$</td>
<td>$10^{-8}$</td>
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</tbody>
</table>

#### OSPOD model reduction error

<table>
<thead>
<tr>
<th>Pod basis rank $\ell$</th>
<th>Error(Exact)</th>
<th>Error(A-priori)</th>
<th>Error(A-posteriori)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2$</td>
<td>$10^4$</td>
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<tr>
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<tr>
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## Numerical experiments

<table>
<thead>
<tr>
<th>method</th>
<th>DoF</th>
<th>CPU time</th>
<th>relative time</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite element system</td>
<td>$N_x = 500$</td>
<td>860.75 sec</td>
<td>100.00%</td>
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<tr>
<td>initial basis</td>
<td>$\ell = 35$</td>
<td>110.77 sec</td>
<td>13.02%</td>
</tr>
<tr>
<td>iterative basis updates</td>
<td>$\ell = 15$</td>
<td>37.41 sec</td>
<td>4.40%</td>
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<tr>
<td>OS-POD basis selection</td>
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<td>18.39 sec</td>
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<td>optimal POD basis</td>
<td>$\ell = 13$</td>
<td>11.48 sec</td>
<td>1.35%</td>
</tr>
</tbody>
</table>
References I


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