Universität Konstanz
Fachbereich Mathematik und Statistik
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9th May 2011

Optimization
Exercises 3

✓ Exercise 9 (5 Points)

(1) Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto f(x_1, x_2) := (x_1 + x_2^2)^2$$

in the point $x_0 = (1, 0)$.

Show that $d := (-1, 1)$ is a direction of descent and find all minimal points of the problem

$$\min_{\alpha > 0} f(x_0 + \alpha d).$$

(2) Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto f(x_1, x_2) := 3x_1^4 - 4x_1^2 x_2 + x_2^2.$$

Prove that $x_0 := (0, 0)$ is a stationary point of f. Show that f, restricted on any line through x_0, has a strict local minimum in x_0. Is x_0 a local minimizer of f?

Let $\mathcal{H}$ denote a $\mathbb{K}$-Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

✓ Exercise 10 (5 Points)

(1) Let $b \in \mathcal{H}$ and $A \in \mathcal{L}_b(\mathcal{H}, \mathcal{H})$, the space of all linear, continuous maps on $\mathcal{H}$.

Show that $x_0 \in \mathcal{H}$ is a minimal point of

$$\varphi : \mathcal{H} \to \mathbb{R}, \qquad x \mapsto \varphi(x) := ||Ax - b||_{\mathcal{H}}$$

if and only if the Gaussian normal equation holds:

$$A^* A x_0 = A^* b.$$

Hereby, A^* denotes the adjoint operator to A, i.e. the following implicitely given operator $A \in \mathcal{L}_b(\mathcal{H}, \mathcal{H})$:

$$\forall x, y \in \mathcal{H} : \langle Ax, y \rangle_{\mathcal{H}} = \langle x, A^* y \rangle_{\mathcal{H}}.$$

(2) Use this characterization to solve the following linear regression problem:

Find parameters $x_1, x_2 \in \mathbb{R}$ such that the corresponding regression line

$$\gamma_x : \mathbb{R} \to \mathbb{R}, \qquad t \mapsto \gamma_x(t) := x_1 + x_2 t$$

approximates the measuring points

t_i	1975	1980	1985	1990	1995
γ_i	30	35	38	42	44

optimally, i.e.

$$(x_1, x_2) = \underset{(y_1, y_2)}{\arg\min} \sum_{i=1}^{5} (\gamma_i - \gamma_y(t_i))^2.$$

Exercise 11

Let $x \in \mathcal{H}$ and F a convex, nonempty, closed subset of $\mathcal{H}$.

Show that there is a unique $y \in F$ such that

$$||x - y||_{\mathcal{H}} = \text{dist}(x, F).$$

Hereby, dist denotes the distance function $\text{dist}(y_0, Y) := \inf_{y \in Y} ||y_0 - y||_{\mathcal{H}}$.

Exercise 12

Let F a nonempty, closed, convex subset of $\mathcal{H}$ and $x_0 \in \mathcal{H}$.

Show that for $x \in F$ the following holds:

$$||x_0 - x||_{\mathcal{H}} = \text{dist}(x_0, F) \qquad \Longleftrightarrow \qquad \forall y \in F : \text{Re}\langle x_0 - x, y - x \rangle_{\mathcal{H}} \leq 0.$$

Deadline: Monday, 16$^{\text{th}}$ May, 8:30 am