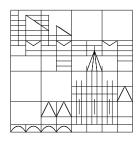
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 $9^{th}$  May 2011

## Optimization Exercises 3

✓ Exercise 9 (5 Points)

(1) Consider the function

$$f: \mathbb{R}^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto f(x_1, x_2) := (x_1 + x_2^2)^2$$

in the point  $x_0 = (1, 0)$ .

Show that d := (-1,1) is a direction of descent and find all minimal points of the problem

$$\min_{\alpha>0} f(x_0 + \alpha d).$$

(2) Consider the function

$$f: \mathbb{R}^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto f(x_1, x_2) := 3x_1^4 - 4x_1^2 x_2 + x_2^2.$$

Prove that  $x_0 := (0,0)$  is a stationary point of f. Show that f, restricted on any line through  $x_0$ , has a strict local minimum in  $x_0$ . Is  $x_0$  a local minimizer of f?

Let  $\mathcal{H}$  denote a  $\mathbb{K}$ -Hilbert space with scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  where  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ .

✓ Exercise 10 (5 Points)

(1) Let  $b \in \mathcal{H}$  and  $A \in \mathcal{L}_b(\mathcal{H}, \mathcal{H})$ , the space of all linear, continuous maps on  $\mathcal{H}$ . Show that  $x_0 \in \mathcal{H}$  is a minimal point of

$$\varphi: \mathcal{H} \to \mathbb{R}, \qquad x \mapsto \varphi(x) := ||Ax - b||_{\mathcal{H}}$$

if and only if the Gaussian normal equation holds:

$$A^*Ax_0 = A^*b.$$

Hereby,  $A^*$  denotes the adjoint operator to A, i.e. the following implicitely given operator  $A \in \mathcal{L}_b(\mathcal{H}, \mathcal{H})$ :

$$\forall x, y \in \mathcal{H} : \langle Ax, y \rangle_{\mathcal{H}} = \langle x, A^*y \rangle_{\mathcal{H}}.$$

(2) Use this characterization to solve the following linear regression problem:

Find parameters  $x_1, x_2 \in \mathbb{R}$  such that the corresponding regression line

$$\gamma_x : \mathbb{R} \to \mathbb{R}, \qquad t \mapsto \gamma_x(t) := x_1 + x_2 t$$

approximates the measuring points

$t_i$	1975	1980	1985	1990	1995
$\gamma_i$	30	35	38	42	44

optimally, i.e.

$$(x_1, x_2) = \underset{(y_1, y_2)}{\operatorname{arg \, min}} \sum_{i=1}^{5} (\gamma_i - \gamma_y(t_i))^2.$$

## Exercise 11

Let  $x \in \mathcal{H}$  and F a convex, nonempty, closed subset of  $\mathcal{H}$ .

Show that there is a unique  $y \in F$  such that

$$||x - y||_{\mathcal{H}} = \operatorname{dist}(x, F).$$

Hereby, dist denotes the distance function  $\operatorname{dist}(y_0, Y) := \inf_{y \in Y} ||y_0 - y||_{\mathcal{H}}$ .

## Exercise 12

Let F a nonempty, closed, convex subset of  $\mathcal{H}$  and  $x_0 \in \mathcal{H}$ .

Show that for  $x \in F$  the following holds:

$$||x_0 - x||_{\mathcal{H}} = \operatorname{dist}(x_0, F) \qquad \Longleftrightarrow \qquad \forall y \in F : \operatorname{Re}\langle x_0 - x, y - x \rangle_{\mathcal{H}} \le 0.$$

**Deadline:** Monday, 16<sup>th</sup> May, 8:30 am