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Fachbereich Mathematik und Statistik
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# Optimization
# Exercises 6

**✓ Exercise 21** (5 Points)

Consider the constrained optimization problem

$$\max_{(x,y)} f(x,y) \qquad \text{subject to} \qquad (x,y) \in F$$

where

$$f(x,y) = x^2 + x^2y^2 + 9y^2 + 9, \qquad F = \{(a,b) \in \mathbb{R}^2 \mid 2a^4 + b^2 \leq 239\}.$$

1. Show that the problem has a global solution.
2. Draw the set of admissible points (you may use MATLAB here).
3. Show that the problem has no inner solution (i.e. no solution in $F^\circ$) and that boundary solutions cannot be unique.
4. Determine the corresponding Lagrange functional and solve the optimization problem.

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**Exercise 22**

A simple strategy to solve the trust-region auxiliary problem (5.4) in the lecture notes approximatively bases on the deepest descent method, respecting the radius $\Delta_k$ for which we trust the model: Consider the optimization problems

$$(1) \quad \begin{cases} \min\limits_{p \in \mathbb{R}^n} f(x_k) + \langle \nabla f(x_k), p \rangle \\ \qquad \text{s.t. } \|p\| \leq \Delta_k \end{cases}, \qquad (2) \quad \begin{cases} \min\limits_{0 \leq t \leq 1} m_k(x_k + tp_k) \\ \quad \text{s.t. } \|tp_k\| \leq \Delta_k \end{cases}$$

where the vector $p_k$ in (2) is the solution of (1). With the solution $t_k$ of (2) we define the Cauchy point $x^{\text{CP}}$ by

$$x_k^{\text{CP}} := x_k + t_k p_k.$$

1. Assume that $x_k$ is no stationary point of $f$. Find the solution $p_k^*$ to the minimization problem (1) – using the Lagrange multiplier method, for example.
2. Show that the solution $t_k^*$ of (2) is given by

$$t_k^* = \begin{cases} 1 & \text{if } \langle \nabla f(x_k), H_k \nabla f(x_k) \rangle \leq 0 \\ \min\left(1, \dfrac{\|\nabla f(x_k)\|^3}{\Delta_k \langle \nabla f(x_k), \nabla^2 f(x_k) \nabla f(x_k) \rangle}\right) & \text{else} \end{cases}.$$

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**Exercise 23**

Another method to solve the trust-region auxiliary problem (5.4) in the lecture notes is the **dogleg strategy**. Hereby, in each iteration step, the following optimization problem is solved instead of (5.4):

$$(3) \quad \begin{cases} \min\limits_{0 \leq t \leq 2} m_k(x_k(t)) \\ \text{s.t. } \|x_k - x_k(t)\| \leq \Delta_k \end{cases}$$

with the piecewise linear path

$$x_k(t) = \begin{cases} x_k + t\left(x_k^{\text{CP}} - x_k\right) & \text{for } 0 \leq t \leq 1 \\ x_k^{\text{CP}} + (t-1)(x_k^{\text{N}} - x_k^{\text{CP}}) & \text{for } 1 \leq t \leq 2 \end{cases}$$

where $x_k^{\text{CP}}$ denotes the normalized Cauchy point

$$x_k^{\text{CP}} = x_k - \frac{||\nabla f(x_k)||^2}{\langle \nabla f(x_k), H \nabla f(x_k) \rangle} \nabla f(x_k)$$

and $x_k^{\text{N}} := x_k - H_k^{-1} \nabla f(x_k)$ is the Newton step.

Hereby, we assume that the approximation of the Hessian matrix $H_k$ is positive definite (which implies $\left\langle x_k^{\text{N}} - x_k^{\text{CP}}, x_k^{\text{CP}} - x_k \right\rangle > 0$; this can be used in the following without proof).

1. Show that the distance function $\|x_k - x_k(t)\|$ increases strictly monotonically in $t$ and that the function of model values $m_k(x_k(t))$ decreases strictly monotonically in $t$.
2. Why are these two abilities helpful by solving the problem (3)?
3. Design a (pseudo-code) algorithm to solve (3).

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**Deadline:** Monday, 27th June, 8:30 am