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Optimierung
5. Übungsblatt

☐ Exercise 15

Let $f : \mathbb{R} \to \mathbb{R}$ a four times differentiable, approximative function – which means that the function values $f(x)$ are not known exactly, but with some error tolerance ϵ_{tol}:

$$f_{\text{known}}(x) = f_{\text{exact}}(x) + \epsilon_{\text{tol}}(x) \qquad \text{where } |\epsilon_{\text{tol}}(x)| \leq \epsilon \text{ for some known } \epsilon > 0.$$

Determine the numerical first derivative $D_c^1(f,h)$ by central differences and prove that the error ϵ_H arising in the numerical approximation of the second derivative $D_c^2(f,h)$ is of the order $\epsilon_H = \mathrm{O}(\epsilon^{\frac{4}{9}})$ for a suitable discretization stepsize h, i.e.

$$||D_c^2(f_{\text{known}}, h) - f''_{\text{exact}}||_\infty < \text{Const} \cdot \epsilon^{\frac{4}{9}}.$$

☐ Exercise 16

Let $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$. Verify the formula

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f(x) = \frac{f(x + \epsilon e_i + \epsilon e_j) - f(x + \epsilon e_i) - f(x + \epsilon e_j) + f(x)}{\epsilon^2} + \mathrm{O}(\epsilon)$$

for the approximation of the Hessian matrix by evaluations of f.

☐ Exercise 17

Let $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ be a quadratic function of the form

$$f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle + \gamma, \quad Q \in \mathbb{R}^{n\times n} \text{ symmetric \& positive definite, } c \in \mathbb{R}^n \text{ and } \gamma \in \mathbb{R}.$$

Let $x_0 \in \mathbb{R}^n$ and H be a symmetric, positive definite matrix. Define $\tilde{f}(x) := f(H^{-\frac{1}{2}}x)$ and $\tilde{x}_0 = H^{\frac{1}{2}}x_0$. Let $(\tilde{x}_k)$ be generated by the Steepest Descent Method with optimal stepsize choice, i.e. $\tilde{x}_{k+1} = \tilde{x}_k + \tilde{t}_k\tilde{d}_k$ where $\tilde{d}_k = -\nabla\tilde{f}(\tilde{x}_k)$ and $\tilde{t}_k = t(\tilde{d}_k)$ as determined in Exercise 6.

Let (x_k) generated by the gradient-like method $x_{k+1} = x_k + t_k d_k$ with optimal stepsize t_k and preconditioner H, i.e. $t_k = t(d_k)$ as in Exercise 6 and $d_k = H^{-1}(-\nabla f(x_k))$.

Show that the two optimization methods are equivalent, i.e. $x^k = H^{-\frac{1}{2}}\tilde{x}_k$ holds for all k, and that the optimal stepsizes are identical: $t_k = \tilde{t}_k$.