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Numerik partieller Differentialgleichungen
1. Übungsblatt

Exercise 1 (4 Points)

Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be a connected and bounded domain with $\mathcal{C}^1$-boundary $\partial\Omega$. Let $D_\psi = \{u \in \mathcal{C}^0(\bar{\Omega}) \mid u = \psi \text{ on } \partial\Omega\}$. We consider the differential operator

$$L : \mathcal{C}^2(\bar{\Omega}) \cap D_0 \to \mathcal{C}^0(\bar{\Omega}), \qquad Lu = -\Delta u.$$

Show:

1. $\langle Lu, v\rangle = \langle u, Lv\rangle$ for $u, v \in \mathcal{C}^2(\bar{\Omega}) \cap D_0$ where $\langle \cdot, \cdot \rangle$ denotes the $\mathcal{L}^2$-scalar product
$$\langle u, v\rangle = \int_\Omega uv \,\mathrm{d}x \qquad (u, v \in \mathcal{C}^2(\bar{\Omega}) \cap D_0).$$
2. L is positive definite, i.e. $\langle Lu, u\rangle > 0$ holds for all $u \in \mathcal{C}^2(\bar{\Omega}) \cap D_0 \backslash \{0\}$.
3. Consider a square domain, e.g. $\Omega = (a, b)^2$. Can the results above be applied in this situation? Give a reference and cite an appropriate theorem.

Exercise 2 (4 Points)

Consider the linear problem

$$\begin{cases} -u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \text{ for all } x \in (0, 1), \\ u(0) = \alpha, u(1) = \beta, \end{cases} \tag{1}$$

where b, c, f are continuous functions on $[0, 1] \subseteq \mathbb{R}$ with $c > 0$ and α, β are real scalars.

1. Discretize (3) using the step size $h = \frac{1}{n+1}$, $n \in \mathbb{N}$, and central difference approximations for $u'(x)$ und $u''(x)$:
$$u'(x) \approx \frac{u(x+h) - u(x-h)}{2h}, \qquad u''(x) \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}.$$
Present the linear system.

2. Show that the coefficient matrix is strictly row diagonal dominant if the stepsize h is small enough.

   What changes if $u'(x)$ is approximated as follows:

$$u'(x) \approx \begin{cases} \dfrac{u(x+h) - u(x)}{h} & \text{if } b(x) < 0, \\ \dfrac{u(x) - u(x-h)}{h} & \text{if } b(x) \geq 0. \end{cases} \tag{2}$$

3. Name an application where the strictly diagonal dominance is important.

Remark: The approximation (2) is called *upwind scheme.* This kind of discretization is often used in the context of convection dominated equations.

Hint: Recall that a matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called *strictly row diagonal dominant* if

$$\sum_{j=1, j\neq i}^{N} |a_{ij}| < |a_{ii}| \qquad \text{for all } i \in \{1, \dots, n\}.$$

Exercise 3 (4 Points)

We consider the following elliptic problem in *divergence form*:

$$\big(a(x)v'(x)\big)' = d(x) \quad \text{for } x \in \Omega = (0,1), \tag{3a}$$

where $a \in \mathcal{C}^1(\overline{\Omega})$ with $a(x) \geq \underline{a} > 0$ for all $x \in \Omega$ and $d \in \mathcal{C}^0(\Omega)$. We state Dirichlet boundary conditions

$$v(0) = \alpha, \quad v(1) = \beta \tag{3b}$$

with $\alpha, \beta \in \mathbb{R}$.

Discretize the interval Ω using the equidistant grid $x_i = ih$, $0 \leq i \leq n+1$, with the mesh size $h = \frac{1}{n+1}$, $n \in \mathbb{N}$. The goal is to obtain a symmetric coefficient matrix $A \in \mathbb{R}^{n \times n}$ when discretizing (3) by finite differences.

1. Approximate the outer derivative $(aw)'$ at the grid point x_i by central difference quotients using the *intermediate* grid points $x_{i\pm\frac{1}{2}} = x_i \pm \frac{h}{2}$, $i = 1, \dots, n$.

   Write $a_i = a(x_i)$, $d_i = d(x_i)$ for $i = 0, \dots, n+1$ and $a_{i\pm\frac{1}{2}} = a(x_i \pm \frac{h}{2})$ for $i = 1, \dots, n$.

   Use the discretization formula

$$w'(x_i) \approx \frac{w_{i+\frac{1}{2}} - w_{i-\frac{1}{2}}}{h}.$$

2. For the inner derivative, replace w by v' and discretize $v'(x_{i\pm\frac{1}{2}})$ by central differences using the grid points x_{i-1}, x_i, x_{i+1} themselves.

   What are the equations that you derive for the approximation of (3)? Compose the coefficient matrix explicitly.