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Numerik partieller Differentialgleichungen
2. Übungsblatt

Exercise 4 (4 Points)

We consider the discretization of the partial differential equation

$$(Lu)(x,y) = f(x,y) \qquad \text{for all } (x,y) \in \Omega = (0,1) \times (0,1) \tag{1a}$$

where

$$(Lu)(x,y) = -\Delta u(x,y) + a(x,y)u_x(x,y) + b(x,y)u_y(x,y) + c(x,y)u(x,y)$$

for all (x,y) on the unit square $\Omega = (0,1) \times (0,1)$ with homogeneous Dirichlet boundary conditions

$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0 \quad \text{for } x,y \in [0,1]. \tag{1b}$$

For general coefficients a, b, c, $f \in C(\bar{\Omega})$ and $c \geq 0$ in Ω, the operator L is not self-adjoint and its discretization is not symmetric.

Discretize (1a) by a five-point centered difference scheme for Δ with n^2 points and mesh width $h = \frac{1}{n+1}$. and by centered difference schemes for the partial derivatives u_x and u_y.

The unknowns are denoted by

$$u_{ij} \approx u(x_i, y_j)$$

where $x_i = ih$ for $i = 1, \ldots, n$.

Compute the coefficient matrix $A \in \mathbb{R}^{n^2 \times n^2}$ and the right-hand side $b \in \mathbb{R}^{n^2}$ so that the discretization of (1) can be formulated as a linear system of the form

$$Au = b.$$

Formulate conditions such that is the matrix A is an L_0-matrix, i.e. $a_{kl} \leq 0$ for all $k \neq l$.

Remark: The L_0 condition is important in the stability analysis for numerical solution methods.

Exercise 5 (4 Points)

Consider the interval (a, b) and $f \in \mathcal{C}^4((a, b), \mathbb{R})$. Let $n \in \mathbb{N}$, $h = \frac{b-a}{n-1}$ and $x_i = a+(i-1)h$ for $i = 1, \ldots, n$.

Show that for the discretization formulae

$$\begin{aligned} D_x^+ f(x_i) &= \frac{f(x_{i+1}) - f(x_i)}{h}; & D_x^0 f(x_i) &= \frac{f(x_{i+1}) - f(x_{i-1})}{2h}; \\ D_x^- f(x_i) &= \frac{f(x_i) - f(x_{i-1})}{h}; & D_{xx} f(x_i) &= \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}; \end{aligned} \quad (2)$$

one has $D_x^+ f - f' = \mathcal{O}(h)$, $D_x^- f - f' = \mathcal{O}(h)$, $D_x^0 f - f' = \mathcal{O}(h^2)$ and $D_{xx} f - f'' = \mathcal{O}(h^2)$.

Exercise 6 (4 Points)

Let A be a block-tridiagonal matrix of the form

$$A = \begin{pmatrix} A_1 & C_1 & 0 & \ldots & 0 \\ B_2 & A_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{n-1} & A_{n-1} & C_{n-1} \\ 0 & \ldots & 0 & B_n & A_n \end{pmatrix},$$

where the A_ls $(1 \leq l \leq n)$ are quadratic matrices of the size m_l. Further, $B_l \in \mathbb{R}^{m_l \times m_{l-1}}$ for $l = 2, \ldots, n$ and $C_l \in \mathbb{R}^{m_l \times m_{l+1}}$ for $l = 1, \ldots, n-1$ hold.

- Derive an algorithm which realizes the factorization

$$A = \begin{pmatrix} D_1 & 0 & 0 & \ldots & 0 \\ B_2 & D_2 & & & \\ & \ddots & \ddots & & \\ & & B_{n-1} & D_{n-1} & 0 \\ 0 & \ldots & 0 & B_n & D_n \end{pmatrix} \begin{pmatrix} E_1 & F_1 & 0 & \ldots & 0 \\ 0 & E_2 & F_2 & & \\ & & \ddots & \ddots & \\ & & & E_{n-1} & F_{n-1} \\ 0 & \ldots & 0 & 0 & E_n \end{pmatrix} =: LU,$$

  where $E_l \in \mathbb{R}^{m_l \times m_l}$ denote identity matrices.

  If it is necessary, suppose the invertibility of certain matrices.

- Assume that the matrices

$$A^{(l)} = \begin{pmatrix} A_1 & C_1 & 0 & \ldots & 0 \\ B_2 & A_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{l-1} & A_{l-1} & C_{l-1} \\ 0 & \ldots & 0 & B_l & A_l \end{pmatrix}, \qquad l = 1, \ldots, n-1,$$

  are non-singular. Show that D_l^{-1} exist for $l = 1, \ldots, n-1$.

Remark: The LU-block decomposition is useful for the iterative solution of $Au = b$.