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Fachbereich Mathematik und Statistik
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Numerik partieller Differentialgleichungen
3. Übungsblatt

Exercise 7 (4 Points)

Let $A \in \mathbb{R}^{M^2 \times M^2}$ be the matrix obtained by the classical finite difference method for solving the boundary value problem

$$\begin{aligned} -\Delta u &= g \quad \text{in } \Omega = (0,1) \times (0,1), &\text{(1a)}\\ u &= \gamma \quad \text{on } \partial\Omega &\text{(1b)} \end{aligned}$$

with stepsize $h = \frac{1}{M+1}$.

Show that the vectors $u^{kl} \in \mathbb{R}^{M^2}$, $(u^{kl})_{ij} = \sin\left(\frac{ik\pi}{M+1}\right) \sin\left(\frac{jl\pi}{M+1}\right)$, are the eigenvectors of A. What are the corresponding eigenvalues λ_{kl}?

Exercise 8 (4 Points)

Let $\Omega \subset \mathbb{R}^2$ a bounded domain with piecewise smooth boundary. Consider the problem

$$\begin{aligned} -\Delta v &= \lambda v \quad \text{in } \Omega, &\text{(2a)}\\ v &= 0 \quad \text{on } \partial\Omega &\text{(2b)} \end{aligned}$$

A solution $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$, $v \neq 0$, is called an eigenfunction to the eigenvalue λ.

1. Show that all eigenvalues λ of (2a) are positive.
2. Let v_1, v_2 be eigenfunctions to the corresponding eigenvalues λ_1, λ_2 with $\lambda_1 \neq \lambda_2$.

   Show that v_1, v_2 are orthogonal with respect to the $\mathcal{L}^2$-scalar product.
3. Let $\Omega = (0,1) \times (0,1)$. Show that the eigenvalues of (2a) are $\lambda_{kl} = \pi^2(k^2 + l^2)$. Compare the corresponding eigenfunctions with those of Exercise 7.
4. Show that the differences between the eigenvalues in Exercise 7 and the corresponding eigenvalues in Exercise 8 are of the order $\mathcal{O}(h^2)$.

Exercise 9 (4 Points)

Consider the elliptic differential equation with Neumann condition on the boundary

$$\begin{aligned} \Delta u(x,y) &= f(x,y) \quad \text{in } \Omega, && \text{(3a)} \\ \frac{\partial u}{\partial \vec{n}} &= g(x,y) \quad \text{on } \Gamma = \partial\Omega && \text{(3b)} \end{aligned}$$

where Ω is a rectangle domain $(0,a)\times(0,b)$. To simplify matters, we consider a uniformly equidistant grid, i.e. we choose grid points (ih, jh) for $i = 0,1,...,M$ and $j = 0,1,...,N$ such that $Mh = a$ and $Nh = b$.

We have to distinguish between four different types of grid points: *inner points* (x_i, y_j) where $i, j \in I \times J = (1,...,M-1) \times (1,...,N-1)$, *boundary points* (x_i, y_j) where either $i \in I$ and $j \in \{0, N\}$ or $i \in \{0, M\}$ and $j \in J$, *corner points* (x_i, y_j) where $i \in \{0, M\}$ and $j \in \{0, N\}$, and so-called *ghost points* (x_i, y_j) where either $i \in I$ and $j \in \{-1, N+1\}$ or $i \in \{-1, M+1\}$ and $j \in J$.

Remark: Ghost points are no "real" grid points, but they appear in the formulation of the finite differences. They can be compensated by plugging in the boundary information.

1. Formulate difference equations for the problem by using the five-point stencil

$$\Delta u(x,y) \approx \frac{u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h) - 4u(x,y)}{h^2}$$

   for all grid points (ih, jh), $i = 0,1,...,M$ and $j = 0,1,...,N$. Here the ghost points will be needed. Note the tacit assumption that the right-hand side f is also defined on Γ.

   For this formulation, approximate the Neumann condition $\frac{\partial u}{\partial \vec{n}}$ on boundary points by central differences:

$$u_x(x,y) \approx \frac{u(x+h,y) - u(x-h,y)}{2h}, \quad u_y(x,y) \approx \frac{u(x,y+h) - u(x,y-h)}{2h}.$$

   At the corner points, where $\vec{n}$ is undefined, approximate the "normal derivative" by the average of the two derivatives along the two outer normals to the sides meeting at the corner (use also central differences).

2. Formulate explicitly the system matrix for $M = N = 2$ and $g \equiv 0$. Here, of course, the ghost points have to be eliminated.

3. Assume again $g \equiv 0$. Show that solutions to the problem cannot be unique. Furthermore, show that this matches with the fact of the non-invertibility of the discretization matrix.

Merry Christmas and a happy new year!