Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Stefan Volkwein
Martin Gubisch, Roberta Mancini, Stefan Trenz
Wintersemester 2011/2012

Ausgabe: 2012/01/13
Abgabe: 2012/01/20

Numerik partieller Differentialgleichungen
4. Übungsblatt

Exercise 10 (FDM for the 1D heat equation) (4 Points)

Let $\Omega = (a, b) \subseteq \mathbb{R}$. Let $T > 0$, $\Theta := (0, T)$, $Q := \Theta \times \Omega$ und $\Sigma := \Theta \times \partial\Omega$.

Consider the linear onedimensional heat equation

$$\begin{cases} y_t(t,x) - \sigma \Delta y(t,x) = f(t,x) & \text{for all } (t,x) \in Q \\ y(t,x) = 0 & \text{for all } (t,x) \in \Sigma \\ y(0,x) = y_0(x) & \text{for all } x \in \Omega \end{cases} \tag{1}$$

with constant coefficient $\sigma > 0$, inhomogeneity $f \in \mathcal{C}^0(Q)$ and initial value $y_0 \in \mathcal{C}^0(\Omega)$.

1. Let $\xi = (x_0, ..., x_{n+1})$ an equidistant discretization of $\bar{\Omega}$. Use the central difference approximation for Δy to approximate (1) by an ordinary system of differential equations for the unknowns $y_j(t) \approx y(t, x_j)$.
2. Write this system in matrix-vector form
$$\dot{Y}(t) + AY(t) = F(t), \qquad Y(0) = Y_0. \tag{2}$$
3. Approximate the time derivatives now by backward differences on an equidistant discretization of $[0, T]$. Determine the linear equations and the matrix-vector form that arise by solving (2) with the implicit Euler method.
4. What disadvantage arises if the time derivatives are approximated by central differences?

Exercise 11 (Weak formulation and finite elements) (4 Points)

Consider the same situation as in Exercise 10.

1. Define the family $(\phi_i)_{i \in I} \subseteq \mathcal{C}^0([a,b], \mathbb{R})$ of hat functions ϕ_i, $I = \{1, ..., n\}$, with $\phi_i(x) = 0$ for $x \notin [x_{i-1}, x_{i+1}]$, $\phi_i(x_i) = 1$ and ϕ_i linear on $[x_{i-1}, x_i]$ and on $[x_i, x_{i+1}]$.

2. Let $z_0, ..., z_{n+1} \in \mathbb{R}$. Define the function $z \in \mathcal{C}^0([a,b],\mathbb{R})$ piecewise linear on the intervals $[x_i, x_{i+1}]$ with $z(x_i) = z_i$.
3. To solve (1) numerically, we make the ansatz

$$\hat{y}(t,x) = \sum_{i=1}^{n} g_i(t)\phi_i(x)$$

for desired time-dependent coefficients $g_i(t)$. Write down the weak formulation for (1) with test functions ϕ_j to set up a system of equations for the coefficient functions.
4. Introduce matrices $\Phi, \Psi \in \mathbb{R}^{n\times n}$ and vectors $g_0, \phi(t) \in \mathbb{R}^n$ such that the equations above can be written in matrix-vector form

$$\begin{cases} \Phi\dot{g}(t) + \sigma\Psi g(t) &= \varphi(t) \\ \Phi g(0) &= g_0. \end{cases} \tag{3}$$

Φ is called the "mass matrix", Ψ the "stiffness matrix". This terminology comes from the classical elasticity theory.

Remark: In this exercise, it is not necessary to calculate the $\mathcal{L}^2$-scalar products arising in the weak formulation explicitely.

Exercise 12 (FEM for the 1D heat equation) (4 Points)

(1) shall be solved now by the Finite Element Method.

1. Calculate the $\mathcal{L}^2$-scalar product

$$\langle \phi_i, z\rangle_{\mathcal{L}^2([a,b],\mathbb{R})} = \int_a^b \phi_i(x)z(x)\mathrm{d}x$$

with the interpolated function z from Exercise 11.

Use this to determine the matrices Φ and Ψ explicitely.
2. Replace y_0 and $f(t)$ by such continuous, piecewise linear functions to approximate g_0 and $\phi(t)$.

Alternatively, approximate the integrals arising for g_0 and $\phi(t)$ by the Simpson rule. What do you observe?
3. Use backward differences on an equidistant discretization for $[0,T]$ and use the implicit Euler method to solve (1) as a system of linear equations. State the coefficient matrices explicitely.
4. Why can't the family $(\phi_i)_{i\in I}$ be applied in the case of inhomogeneous Dirichlet boundary conditions $y(t,a) = y_a(t)$, $y(t,b) = y_b(t)$ with non-zero boundary functions $y_a, y_b \in \mathcal{C}^0([0,T],\mathbb{R})$?