Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Stefan Volkwein
Martin Gubisch, Roberta Mancini, Stefan Trenz
Wintersemester 2011/2012

Ausgabe: 2012/01/20
Abgabe: 2012/01/27

Numerik partieller Differentialgleichungen
5. Übungsblatt

Exercise 13 (Quadratic ansatz functions) (4 Points)

Let $\Omega = (a,b) \subseteq \mathbb{R}$ and $x_0, ..., x_{n+1} \in \bar{\Omega}$ with $x_0 = a,\ x_{n+1} = b$, $x_i < x_{i+1}$.

1. Define piecewise quadratic ansatz functions ϕ_i, $i = 1, ..., n$, and $\phi_{i+\frac{1}{2}}$, $i = 0, ..., n$, in $\mathcal{C}^0(\bar{\Omega})$ such that

$$\phi_i(x_j) = \delta_{ij} \qquad \& \qquad \phi_i(x_{j+\frac{1}{2}}) = 0,$$
$$\phi_{i+\frac{1}{2}}(x_j) = 0 \qquad \& \qquad \phi_{i+\frac{1}{2}}(x_{j+\frac{1}{2}}) = \delta_{ij}.$$

   Hint: A quadratic function is determined uniquely by its values on three interpolation points.

2. Calculate the derivatives $\phi_i, \phi_{i+\frac{1}{2}}$.
3. Draw $\phi_i, \phi_{i+\frac{1}{2}}, \phi_i', \phi_{i+\frac{1}{2}}'$ for one fixed i in different colours in one **large** plot. Don't forget to label your axes.
4. Consider the ansatz

$$y(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x) + \sum_{i=0}^{n} \alpha_{i+\frac{1}{2}} \phi_{i+\frac{1}{2}}(x)$$

   for arbitrary coefficients $\alpha_i, \alpha_{i+\frac{1}{2}} \in \mathbb{R}$. Explain why y in general is just of $\mathcal{H}^1$-regularity (you don't need to prove this).

Exercise 14 (Weak derivatives) (4 Points)

Let $\Omega = (-1, 1)$.

1. Let $u \in \mathcal{L}^2(\Omega)$ Define: $v \in \mathcal{L}^2(\Omega)$ is the weak derivative of u.

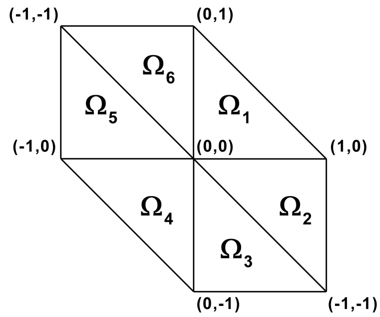
2. $u(x) = \text{abs}(x)$. Show that the weak derivative $v = u' \in \mathcal{L}^2(\Omega)$ of u is given by

$$u'(x) = \begin{cases} -1 & x \in [-1, 0] \\ 1 & x \in [0, 1] \end{cases} .$$

3. Prove that u' is not weakly differentiable in $\mathcal{L}^2(\Omega)$.

Exercise 15 (Ansatz functions in two dimensions) (4 Points)

Let $\Omega = (-1, 1) \times (-1, 1)$. Consider the following triangulation of Ω:



1. Define the function $S : \bar{\Omega} \to \mathbb{R}$ describing the canonical simplex on Ω, i.e. $S_i = S|_{\bar{\Omega}_i}$ is a plain with $S_i(0, 0) = 1$ and $S_i(x, y) = 0$ for all other grid points $(x, y) \in \bar{\Omega}$.
2. Draw the graph of S.
3. Assume that Ω is triangulized by n^2 inner grid points x_i, y_j now. Let $f \in \mathcal{C}^0(\bar{\Omega})$, $f_{ij} = f(x_i, y_j)$ and

$$F(x, y) = \sum_{i,j=1}^{n} f_{ij}\phi_{ij}(x, y)$$

   where ϕ_{ij} is the simplex with centre (x_i, y_j).

   Describe the graph of F.
4. Is F a good approximation for f?