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Numerik partieller Differentialgleichungen
2. Programm

Program 2 (FDM and FEM for the 1D heat equation) (8 Points)

Let $\Omega = (a, b) \subseteq \mathbb{R}$, $T > 0$, $\Theta = (0, T)$, $Q = \Theta \times \Omega$ and $\Sigma = \Theta \times \partial\Omega$. Further, let $\sigma > 0$ and $f \in \mathcal{C}^0(\bar{Q}, \mathbb{R})$, $y_0 \in \mathcal{C}^0(\bar{\Omega}, \mathbb{R})$.

Consider the linear heat equation

$$\begin{cases} y_t(t,x) - \sigma\Delta y(t,x) &= f(t,x) & \text{for all } (t,x) \in Q \\ y(t,x) &= 0 & \text{for all } (t,x) \in \Sigma \\ y(0,x) &= y_0(x) & \text{for all } x \in \Omega \end{cases} . \tag{1}$$

1. Solve (1) numerically with FDM. Herefore, use the discretizations from exercise 10.

   Write a function `fdm_parabolic_1D(a, b, T, sigma, f, y0)` which is called from a `main.m` file where $\texttt{f} \in \mathbb{R}^{m \times n}$ and $\texttt{y0} \in \mathbb{R}^{1 \times n}$ are the discretizations of f or y_0, respectively, for the equidistant representations $x = (x_1, ..., x_n)$, $x_1 = a$, $x_n = b$ and $t = t_1, ..., t_m$, $t_1 = 0$, $t_m = T$. The output is an $m \times n$ matrix `y` with $\texttt{y}_{ij} \approx y(t_i, x_j)$.

2. Solve (1) numerically with FEM. Use the discretizations from exercise 12 here.

   Your solver function `fem_parabolic_1D` shall have the same input and output arguments as in the previous part.

3. Test your programs with the data $[a, b] = [0, 1]$, $T = 10$, $\sigma = 1$, $y_0 = 0$ and $f(t, x) = 2t\sin(\pi x) + \pi^2 \sin(\pi x)t^2$. Use $m = 250$, $n = 500$. Plot the solution on the time-space grid of Q. Notice that using sparse matrices and avoiding unnecessary loops speeds up the running time of the program and reduces the needed processor memory essentially.

4. Compute the exact solution y by hand and calculate the maximal errors on the time-space grid between the numerical and the exact solutions for $m = 250$ and $n = 5, 10, 15, 20, 25, 30, 40, 50, 65, 80, 100$. Show with a suitable plot (logarithmic scales may be helpful) that the errors are of the order $\mathcal{O}(h^2)$ where $h = \frac{1}{n-1}$. Repeat this with $m = 25$ and explain the difference between the plots.

5. Let $T = 1$, $f = 0$ and $y_0 = 1$. Choose $m = 25$ and $n = 50$ (for example) and plot the numerical solution $(t, x) \mapsto y(t, x)$. Why is y discontinuos although the data functions are $\mathcal{C}^\infty$?