

Kinetic derivation of a finite difference scheme for the incompressible Navier Stokes equation ^{*}

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Abstract

In the present paper the low Mach number limit of kinetic equations is used to develop a discretization for the incompressible Navier-Stokes equation. The kinetic equation is discretized with a first and second order discretization in space. The discretized equation is then considered in the low Mach number limit. Using this limit a second order discretization for the convective part in the incompressible Navier Stokes equation is obtained. Numerical experiments are shown comparing different approaches.

Keywords. kinetic equations, asymptotic analysis, low Mach number limit, second order upwind discretization, slope limiter, incompressible Navier-Stokes equation

AMS subject classifications.

1 Introduction

Kinetic equations or discrete velocity models of kinetic equations yield in the limit for small Knudsen or Mach numbers an approximation of macroscopic equations like the Euler or incompressible Navier Stokes equation. Discretizations of kinetic models are often used in combination with the above limiting procedures to develop discretizations for the corresponding macroscopic limit equation.

We consider first the Euler or hydrodynamic limit: As a first example of the above approach we mention the Kinetic Schemes. A simple kinetic relaxation model with a so called BGK operator is discretized. In the limit a scheme for the hyperbolic limit equation is obtained. There is a large amount of literature on the subject, examples can be found in [21, 20, 6, 7, 15].

A second example is given by the relaxation schemes developed in [14]. Based on a simplified kinetic model efficient second order upwind discretizations for the Euler equation are developed. For further examples and related schemes, see [3, 19, 12, 18].

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As already mentioned, another scaling - the diffusive scaling - gives in the limit the incompressible Navier Stokes equation. Again kinetic models - usually based on a small number of velocities - are discretized and investigated in the incompressible Navier Stokes limit.

A well known example are the Lattice-Boltzmann methods, see [9, 4, 11, 8]. Also relaxation schemes for diffusive limits have been developed for example in [13, 17]. See also [18, 1] for related schemes for the limit equations.

In the present paper we start by recalling the diffusive scaling of kinetic equations, leading to the incompressible Navier Stokes equation. Then, a natural discretization of the kinetic equation is used to obtain in the limit a second order slope limiting procedure for the convective term of the Navier Stokes equation. For the slope limiting procedure the behavior of the solution in all spatial directions is important not only along the values of the solution along the coordinate axis.

The paper is organized as follows: Section 2 contains a short description of the results of the asymptotic analysis leading from kinetic equations to the incompressible Navier Stokes equation. In section 3 the asymptotic procedure is performed for the discretized kinetic equations and a general limit discretization for the incompressible Navier Stokes equation is derived. In 4 we concentrate on the derivation of the discretization of the convective part. A first and second order upwind discretization for the limit equation is derived. Whereas the first order discretization is standard, the second order discretization includes a multidimensional slope limiting procedure.

2 Kinetic Equations and the Incompressible Navier Stokes Equation

The incompressible Navier Stokes equation

$$\partial_t u + u \cdot \nabla u + \nabla_x p = \mu \Delta_x u, \quad \operatorname{div}_x u = 0 \quad (2.1)$$

is a reasonable model to describe the large scale and long time behavior of a slowly flowing isothermal gas. We will use the fact that (2.1) can be formally obtained as scaling limit of a Boltzmann type kinetic equation

$$\partial_t f + v \cdot \nabla_x f = J(f). \quad (2.2)$$

Here, $f = f(x, v, t)$ is the phase space density of the gas atoms which we consider, for simplicity, in the two dimensional case $x = (x_1, x_2) \in \mathbb{R}^2$, $v = (v_1, v_2) \in \mathbb{R}^2$. We will not specify the complete structure of the collision operator $J(f)$. Only those properties which are important in the Navier Stokes limit will be listed below.

Using the diffusive space-time scaling $x \mapsto x/\epsilon$, and $t \mapsto t/\epsilon^2$, the restriction to large scale and long time behavior is incorporated. We find the scaled kinetic equation

$$\partial_t f + \frac{1}{\epsilon} v \cdot \nabla_x f = \frac{1}{\epsilon^2} J(f). \quad (2.3)$$

The assumption of very slow, isothermal flows which exhibit only small density variations is then taken care of by assuming that f is only a small perturbation of the Maxwellian velocity distribution M

$$M(v) = \frac{1}{2\pi} \exp\left(-\frac{|v|^2}{2}\right), \quad v \in \mathbb{R}^2$$

which corresponds to the constant state: density one, velocity zero, and temperature one. Our precise assumption on the structure of f is

$$f = M(1 + \epsilon g), \quad g = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \dots \quad (2.4)$$

The scaled equation (2.3) together with (2.4) is the standard perturbation procedure to obtain (2.1) as limiting problem (see [2, 5, 22]). In the next section, we demonstrate this procedure in a slightly more general situation where (2.3) is modified by adding a diffusive term $D_h(v)f$ and replacing ∇_x with an approximation ∇_x^h .

Let us now list some properties of J which will be needed for the analysis: if (2.4) is inserted into (2.3) we need a Taylor expansion of $J(M + \epsilon M g)$ to compare terms of equal order in ϵ . We have

$$\frac{1}{M} J(M + \epsilon M g) = \epsilon L g + \frac{1}{2} \epsilon^2 Q(g, g) + \epsilon^3 R(g) \quad (2.5)$$

where L involves the first and Q the second Frechet derivative of J at the point M (see [2] for details). The exact structure of the remainder R is not relevant in the limit. Note that the zero order term in (2.5) drops out because of the assumed equilibrium condition

$$J(M) = 0. \quad (2.6)$$

Another important assumption is that the collision invariants of J are the functions $1, v_1, v_2$ (in isothermal flows $|v|^2$ is not a collision invariant), which means in terms of the weighted \mathbb{L}^2 scalar product $\langle g, h \rangle = \int_{\mathbb{R}^2} g h M dv$

$$\left\langle \frac{1}{M} J(f), \psi \right\rangle = 0, \quad \psi \in \{1, v_1, v_2\}. \quad (2.7)$$

Note that (2.7) implies together with (2.5) that also

$$\langle Lg, \psi \rangle = \langle Q(g, g), \psi \rangle = \langle R(g), \psi \rangle = 0 \quad (2.8)$$

for all collision invariants ψ . Important assumptions on the operator L are

- 1) L is selfadjoint with respect to $\langle \cdot, \cdot \rangle$ and $\langle Lh, h \rangle \leq 0$.
- 2) L satisfies a Fredholm alternative with a three dimensional kernel spanned by the collision invariants (we denote the pseudo inverse of L by L^\dagger , i.e. $L^\dagger L$ is the orthogonal projection onto $(\ker L)^\perp$).

Additionally, with $\mu, \lambda > 0$

$$\langle L^\dagger(v_k v_l), v_i \rangle = 0, \quad \langle L^\dagger(v_k v_l), v_i v_j \rangle = -\lambda \delta_{kl} \delta_{ij} - \mu(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (2.9)$$

which would be a consequence of invariance of J under certain coordinate transformations in the velocity space as well as L being self adjoint and semi definite. Finally, we need a property of Q which is a direct consequence of the relation

$$Q(h, h) = -Lh^2 \quad \text{for } h = \alpha + \beta \cdot v \quad (2.10)$$

(see [2] for the derivation). Using the fact that $1, v_1, v_2$ are in the kernel of L and that $L^\dagger L$ is the projection onto $(\ker L)^\perp$, we conclude

$$-L^\dagger Q(h, h) = \beta_i \beta_j L^\dagger L(v_i v_j) = \beta_i \beta_j (v_i v_j - \delta_{ij}). \quad (2.11)$$

The projection has been calculated with the usual Schmidt procedure which requires the scalar products $\langle 1, v_i v_j \rangle$ and $\langle v_k, v_i v_j \rangle$. Such moments of the standard Maxwellian will be frequently used later and we list them here for convenience

$$\begin{aligned} \langle 1, 1 \rangle &= 1, & \langle 1, v_i \rangle &= 0, & \langle v_i, v_j \rangle &= \delta_{ij} \\ \langle v_i v_j, v_k \rangle &= 1, & \langle v_i v_j, v_k v_l \rangle &= \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}. \end{aligned} \quad (2.12)$$

3 The discretized kinetic equation and derivation of macroscopic discretization

We start with the kinetic equation (2.2) which is discretized using the method of lines

$$\partial_t f + v \cdot \nabla_x^h f - D_h(v) f = J(f).$$

In this section, we only assume that $\nabla_x^h = (\partial_{x_1}^h, \partial_{x_2}^h)$ and $D_h(v)$ are linear operators, that ∇_x^h is independent of v , and that the components of ∇_x^h commute. In the next section we choose $\partial_{x_i}^h$ as central difference approximations and $D_h(v)$ as numerical viscosity term.

Introducing the diffusive scaling as in the previous section and rescaling $D_h(v) f$ according to $\epsilon^2 D_h(v) f$, we find

$$\partial_t f + \frac{1}{\epsilon} v \cdot \nabla_x^h f - D_h(v) f = \frac{1}{\epsilon^2} J(f).$$

With the expansion (2.4) and (2.5), we then get

$$\partial_t g + \frac{1}{\epsilon} v \cdot \nabla_x^h g - D_h(v) g = \frac{1}{\epsilon^2} Lg + \frac{1}{2\epsilon} Q(g, g) + R(g).$$

and the regular expansion $g = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \dots$ leads to

$$\begin{aligned} &\partial_t g + \frac{1}{\epsilon} v \cdot \nabla_x^h g - D_h(v) g \\ &= \frac{1}{\epsilon^2} Lg_0 + \frac{1}{\epsilon} \left(Lg_1 + \frac{1}{2} Q(g_0, g_0) \right) + \left(Lg_2 + Q(g_0, g_1) + R(g_0) \right) + \mathcal{O}(\epsilon) \end{aligned} \quad (3.1)$$

In lowest order ϵ^{-2} we immediately find $Lg_0 = 0$ so that $g_0 \in \ker L$, i.e.

$$g_0(v) = \rho + u \cdot v, \quad \rho \in \mathbb{R}, u \in \mathbb{R}^2 \quad (3.2)$$

with parameters ρ, u which are yet undetermined. In order ϵ^{-1} , we have

$$v \cdot \nabla_x^h g_0 = Lg_1 + \frac{1}{2}Q(g_0, g_0) \quad (3.3)$$

and with (2.8) we conclude

$$\langle v \cdot \nabla_x^h g_0, 1 \rangle = 0, \quad \langle v \cdot \nabla_x^h g_0, v_i \rangle = 0. \quad (3.4)$$

Using (3.2), the first condition can be reformulated

$$\begin{aligned} 0 &= \partial_{x_i}^h \langle g_0, v_i \rangle = \partial_{x_i}^h \langle \rho + v_j u_j, v_i \rangle \\ &= \langle 1, v_i \rangle \partial_{x_i}^h \rho + \langle v_j, v_i \rangle \partial_{x_i}^h u_j = \partial_{x_i}^h u_i =: \operatorname{div}_x^h u \end{aligned}$$

where we have used moment relations from (2.12). Similarly, we find

$$0 = \partial_{x_j}^h \langle v_j g_0, v_i \rangle = \langle v_j, v_i \rangle \partial_{x_i}^h \rho + \langle v_j v_k, v_i \rangle \partial_{x_i}^h u_k = \partial_{x_i}^h \rho$$

so that (3.4) implies

$$\operatorname{div}_x^h u = 0, \quad \nabla_x^h \rho = 0 \quad (3.5)$$

which has to be satisfied by the parameters ρ, u in (3.2). Applying L^\dagger to (3.3), we obtain the projection of g_1 onto $(\ker L)^\perp$, so that with additional parameters $\rho^{(1)}, u^{(1)}$

$$g_1 = L^\dagger \left(v \cdot \nabla_x^h g_0 - \frac{1}{2}Q(g_0, g_0) \right) + \rho^{(1)} + u^{(1)} \cdot v.$$

Using (3.5), we can calculate

$$v \cdot \nabla_x^h g_0 = v_i \partial_{x_i}^h (\rho + v_j u_j) = v_i v_j \partial_{x_i}^h u_j$$

and hence $L^\dagger v \cdot \nabla_x^h g_0 = (\partial_{x_i}^h u_j) L^\dagger (v_i v_j)$. The expression $-L^\dagger Q(g_0, g_0)$ can be simplified with (2.11), so that

$$g_1 = (\partial_{x_i}^h u_j) L^\dagger (v_i v_j) + \frac{1}{2} u_i u_j (v_i v_j - \delta_{ij}) + \rho^{(1)} + u_j^{(1)} v_j. \quad (3.6)$$

Going back to (3.1) and collecting terms of order ϵ^0 , we find

$$\partial_t g_0 + v \cdot \nabla_x^h g_1 - D_h(v) g_0 = Lg_2 + Q(g_0, g_1) + R(g_0). \quad (3.7)$$

Again, property (2.8) yields conditions on the undetermined parameters in g_0 and g_1 . Integrating (3.7) and observing that $\langle g_1, v_i \rangle = u_i^{(1)}$ because of (2.9) and (2.12), we get a divergence condition on $u^{(1)}$

$$\partial_t \rho + \operatorname{div}_x^h u^{(1)} = \langle D_h(v) g_0, 1 \rangle.$$

Integration of $v \cdot \nabla_x^h g_1$ after multiplication with $\psi = v_j$ yields with the help of (2.9) and (2.12)

$$\begin{aligned} \partial_{x_i}^h \langle v_i g_1, v_j \rangle &= (\partial_{x_i}^h \partial_{x_k}^h u_l) (-\lambda \delta_{kl} \delta_{ij} - \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})) \\ &\quad + \frac{1}{2} \partial_{x_i}^h (u_l u_k) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \partial_{x_j}^h \rho^{(1)} \end{aligned}$$

so that the v_j weighted integral of (3.7) leads to

$$\begin{aligned} \partial_t u_j + \partial_{x_i}^h (u_i u_j) - \langle D_h(v) g_0, v_j \rangle + \partial_{x_j}^h \rho^{(1)} \\ = \lambda \partial_{x_j}^h (\partial_{x_k}^h u_k) + \mu \partial_{x_k}^h (\partial_{x_j}^h u_k) + \mu \partial_{x_i}^h \partial_{x_i}^h u_j. \end{aligned}$$

Taking into account that $\partial_{x_1}^h$ and $\partial_{x_2}^h$ commute, relation (3.5) implies that $\lambda \partial_{x_j}^h (\partial_{x_k}^h u_k) = \mu \partial_{x_k}^h (\partial_{x_j}^h u_k) = 0$. Introducing the discretized Laplacian $\Delta_x^h = \partial_{x_i}^h \partial_{x_i}^h$, we have the final result

$$\partial_t u_j + \partial_{x_i}^h (u_i u_j) - \langle D_h(v) g_0, v_j \rangle + \partial_{x_j}^h \rho^{(1)} = \mu \Delta_x^h u_j, \quad \operatorname{div}_x^h u = 0. \quad (3.8)$$

Note that (3.8) reduces to the incompressible Navier Stokes equation (2.1) if we choose $\nabla_x^h = \nabla_x$ and $D_h(v) = 0$. Obviously, $\rho^{(1)}$ takes the role of the pressure and $\partial_{x_i}^h (u_i u_j) - \langle D_h(v) g_0, v_j \rangle$ gives a discretization of the convective terms ones the numerical viscosity is fixed.

4 First and second order upwind schemes

To find expressions for ∇_x^h and the numerical viscosity $D_h(v)$ we consider the linear transport part of the kinetic equation in two dimensions:

$$v \cdot \nabla_x f = v_1 \partial_{x_1} f + v_2 \partial_{x_2} f. \quad (4.1)$$

A *first order* discretization is given by

$$v_1 \partial_{x_1}^h f + v_2 \partial_{x_2}^h f - \frac{c_1 h}{2} \partial_{x_1}^{2,h} f - \frac{c_2 h}{2} \partial_{x_2}^{2,h} f \quad (4.2)$$

with positive constants c_1, c_2 and

$$\begin{aligned} (\partial_{x_1}^h f)_{ij} &= \frac{1}{2h} (f_{i+1j} - f_{i-1j}), & (\partial_{x_1}^{2,h} f)_{ij} &= \frac{1}{h^2} (f_{i+1j} - 2f_{ij} + f_{i-1j}), \\ (\partial_{x_2}^h f)_{ij} &= \frac{1}{2h} (f_{ij+1} - f_{ij-1}), & (\partial_{x_2}^{2,h} f)_{ij} &= \frac{1}{h^2} (f_{ij+1} - 2f_{ij} + f_{ij-1}). \end{aligned}$$

The constants are chosen later according to the macroscopic flow situation under consideration. In view of (4.2), we define

$$D_h(v) f = \left(c_1 \frac{h}{2} \partial_{x_1}^{2,h} f + c_2 \frac{h}{2} \partial_{x_2}^{2,h} f \right)$$

and obtain

$$D_h(v) g_0 = c_1 \partial_{x_1}^{2,h} \rho \frac{h}{2} + c_2 \partial_{x_2}^{2,h} \rho \frac{h}{2} + c_1 \partial_{x_1}^{2,h} u_i \frac{h v_i}{2} + c_2 \partial_{x_2}^{2,h} u_i \frac{h v_i}{2}$$

which yields with (2.12) the required expressions in (3.8)

$$\langle D_h(v) g_0, v \rangle = \frac{h}{2} \left(c_1 \partial_{x_1}^{2,h} u + c_2 \partial_{x_2}^{2,h} u \right).$$

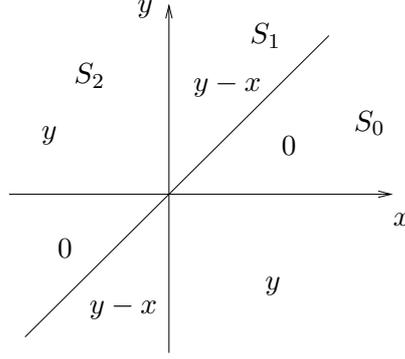


Figure 1: Piecewise linear definition of $\phi(x, y)$ in the sets S_0, S_1, S_2

A *second order* discretization of (4.1) is obtained by slope limiting

$$(v \cdot \nabla_x^h f)_{ij} - \left[\frac{c_1(i, j)}{2h} \left((1 - \varphi_{i+\frac{1}{2}j}) \Delta_{i+\frac{1}{2}j} f - (1 - \varphi_{i-\frac{1}{2}j}) \Delta_{i-\frac{1}{2}j} f \right) + \frac{c_2(i, j)}{2h} \left((1 - \varphi_{ij+\frac{1}{2}}) \Delta_{ij+\frac{1}{2}} f - (1 - \varphi_{ij-\frac{1}{2}}) \Delta_{ij-\frac{1}{2}} f \right) \right] \quad (4.3)$$

where ∇_x^h are again central differences, the f increments are defined by

$$\Delta_{i+\frac{1}{2}j} f = f_{i+1j} - f_{ij}, \quad \Delta_{ij+\frac{1}{2}} f = f_{ij+1} - f_{ij},$$

and

$$\begin{aligned} \varphi_{i+\frac{1}{2}j} &= \varphi(r_{i+\frac{1}{2}j}), & r_{i+\frac{1}{2}j} &= \Delta_{i-\frac{1}{2}j} f / \Delta_{i+\frac{1}{2}j} f \\ \varphi_{ij+\frac{1}{2}} &= \varphi(r_{ij+\frac{1}{2}}), & r_{ij+\frac{1}{2}} &= \Delta_{ij-\frac{1}{2}} f / \Delta_{ij+\frac{1}{2}} f, \end{aligned}$$

with $\varphi(r) = \max\{0, \min\{r, 1\}\}$ being the minmod limiter. Using the definition of φ , one can write expressions like $(1 - \varphi_{i+\frac{1}{2}j}) \Delta_{i+\frac{1}{2}j} f$ as $\phi(\Delta_{i-\frac{1}{2}j} f, \Delta_{i+\frac{1}{2}j} f)$ where ϕ is a continuous, piecewise linear function on \mathbb{R}^2 defined according to figure 1. Extracting the viscosity term in (4.3), we get

$$\begin{aligned} D_h(v) f_{ij} &= \frac{c_1(i, j)}{2h} \left(\phi(\Delta_{i-\frac{1}{2}j} f, \Delta_{i+\frac{1}{2}j} f) - \phi(\Delta_{i-\frac{3}{2}j} f, \Delta_{i-\frac{1}{2}j} f) \right) \\ &+ \frac{c_2(i, j)}{2h} \left(\phi(\Delta_{ij-\frac{1}{2}} f, \Delta_{ij+\frac{1}{2}} f) - \phi(\Delta_{ij-\frac{3}{2}} f, \Delta_{ij-\frac{1}{2}} f) \right) \end{aligned}$$

Now, the task to calculate $\langle D_h(v) g_0, v_j \rangle$ is more involved than in the first order case. We start with the observation that

$$\Delta_{i+\frac{1}{2}j} g_0 = (\Delta_{i+\frac{1}{2}j} u) \cdot v$$

because ρ satisfies $\nabla_x^h \rho = 0$. Hence, a typical term appearing in $\langle D_h(v) g_0, v \rangle$ has the form

$$\langle \phi(\delta_1 \cdot v, \delta_2 \cdot v), v \rangle \quad \delta_1, \delta_2 \in \mathbb{R}^2. \quad (4.4)$$

Introducing the linear map

$$T = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}$$

we can rewrite (4.4) as $\langle \phi(Tv), v \rangle$. Note that ϕ is linear in the convex sets S_0, S_1, S_2 (see figure 1). With the unit vectors $e_1 = (1, 0)$, $e_2 = (0, 1)$ and

$$S(a, b) = S^+(a, b) \cup S^-(a, b), \quad S^\pm = \{\pm(\lambda_1 a + \lambda_2 b) : \lambda_1, \lambda_2 \geq 0\},$$

we can describe these sets as

$$S_0 = S(e_1, e_1 + e_2), \quad S_1 = S(e_1 + e_2, e_2), \quad S_2 = S(e_2, -e_1).$$

Assuming that T is invertible, we conclude that $\phi \circ T$ is linear on the sets $\hat{S}_i = T^{-1}S_i$. Since $S(a, b) = cS(a, b)$ for all $c \neq 0$, we have $\hat{S}_i = \hat{T}(S_i)$ where

$$\hat{T} = (\det T)T^{-1} = \begin{pmatrix} \delta_{22} & -\delta_{12} \\ -\delta_{21} & \delta_{11} \end{pmatrix} = (-\delta_2^\perp \quad \delta_1^\perp).$$

Hence,

$$\hat{S}_0 = S(-\delta_2^\perp, \delta_1^\perp - \delta_2^\perp), \quad \hat{S}_1 = S(\delta_1^\perp - \delta_2^\perp, \delta_1^\perp), \quad \hat{S}_2 = S(\delta_1^\perp, \delta_2^\perp).$$

Taking into account that ϕ vanishes on S_0 , we find

$$\langle \phi(Tv), v \rangle = \int_{\hat{S}_1} (\delta_2 \cdot v - \delta_1 \cdot v)vM(v) dv + \int_{\hat{S}_2} (\delta_2 \cdot v)vM(v) dv$$

or with the help of the matrix valued function

$$I(a, b) = \int_{S(a, b)} v \otimes vM(v) dv$$

that

$$\langle \phi(Tv), v \rangle = I(\delta_1^\perp - \delta_2^\perp, \delta_1^\perp)(\delta_2 - \delta_1) + I(\delta_1^\perp, \delta_2^\perp)\delta_2 = : L(\delta_1, \delta_2).$$

Hence, the numerical viscosity for the second order discretization has the form

$$\begin{aligned} \langle D_h(v)g_0, v \rangle_{ij} &= \frac{c_1(i, j)}{2h} \left(L(\Delta_{i-\frac{1}{2}j}u, \Delta_{i+\frac{1}{2}j}u) - L(\Delta_{i-\frac{3}{2}j}u, \Delta_{i-\frac{1}{2}j}u) \right) \\ &\quad + \frac{c_2(i, j)}{2h} \left(L(\Delta_{ij-\frac{1}{2}}u, \Delta_{ij+\frac{1}{2}}u) - L(\Delta_{ij-\frac{3}{2}}u, \Delta_{ij-\frac{1}{2}}u) \right) \end{aligned}$$

We conclude by giving an explicit formula for the function I . First, we note that, using the symmetry of $M(v)$ and $v \otimes v$

$$I(a, b) = 2 \int_{S^+(a, b)} v \otimes vM(v) dv.$$

Using that $S^+(a, b)$ is a cone with some opening angle $0 < \beta < \pi$ around the

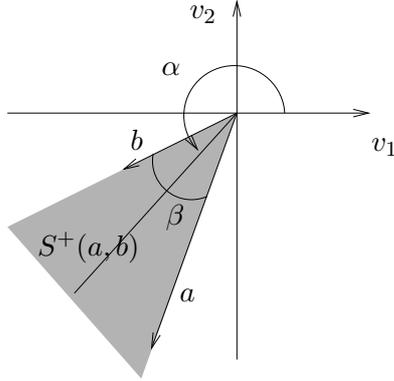


Figure 2: Angles α, β characterizing the cone $S^+(a, b)$

ray in direction α (see figure 2), we go over to polar coordinates and find

$$I(a, b) = 2 \int_{\alpha-\beta/2}^{\alpha+\beta/2} \begin{pmatrix} \cos^2 \psi & \sin \psi \cos \psi \\ \sin \psi \cos \psi & \sin^2 \psi \end{pmatrix} d\psi \int_0^\infty \frac{r^2}{2\pi} e^{-\frac{r^2}{2}} r dr.$$

After some straight forward calculations we get

$$I(a, b) = \frac{1}{\pi} \left(\delta + \sin \delta \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix} \right).$$

In the case where T is not invertible, one can either slightly modify the row vectors δ_1, δ_2 in order to obtain an invertible mapping (note that $\langle \phi(Tv), v \rangle$ is continuous in T), or one can use the relation

$$\langle \phi(Tv), v_j \rangle = \phi(Te_j). \quad (4.5)$$

To prove (4.5), let us consider the case $\delta_2 = \gamma\delta_1$. Then, $Tv = \delta_1 \cdot v \begin{pmatrix} 1 \\ \gamma \end{pmatrix}$ and $\phi(Tv) = (\delta_1 \cdot v)\phi(1, \gamma)$. Now, $\langle \delta_1 \cdot v, v_j \rangle = \delta_1 \cdot e_j$ so that (4.5) follows. The case $\delta_1 = \gamma\delta_2$ can be treated similarly.

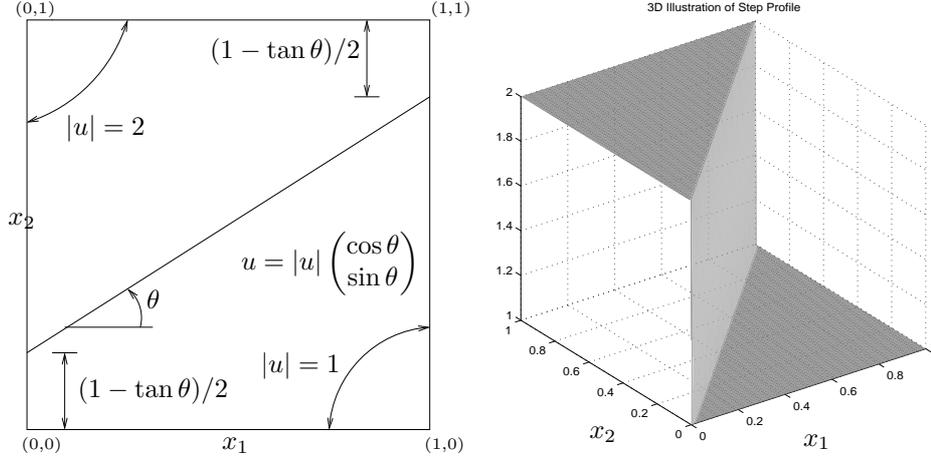
5 Numerical results

We investigate the above first and second order discretizations of the convective term numerically for the following stationary equation and compare them with the second order scheme described in [18]. The viscous term $\varepsilon\Delta u$ is treated by straightforward second order central differences.

$$\nabla_x(u \otimes u) - \varepsilon\Delta u = 0 \quad + \text{boundary conditions}$$

To test our scheme for pure convection we use test examples for the stationary case as presented in [10].

The first example is pure convection of a Step Profile, i.e. the following flow situation is considered: Consider (x_1, x_2) in $[0, 1]^2$. The domain of computation is divided into two subdomains which give a step profile as sketched below.



Boundary values are chosen according to the respective domain. The solution domain is discretized using a 41×41 regular mesh for different flow angles θ . We may choose c_1 and c_2 constant proportional to the maximal flow velocity:

$$c_1 = \max_{ij} \{ |2u_{ij}| \}, \quad c_2 = \max_{ij} \{ |2v_{ij}| \}.$$

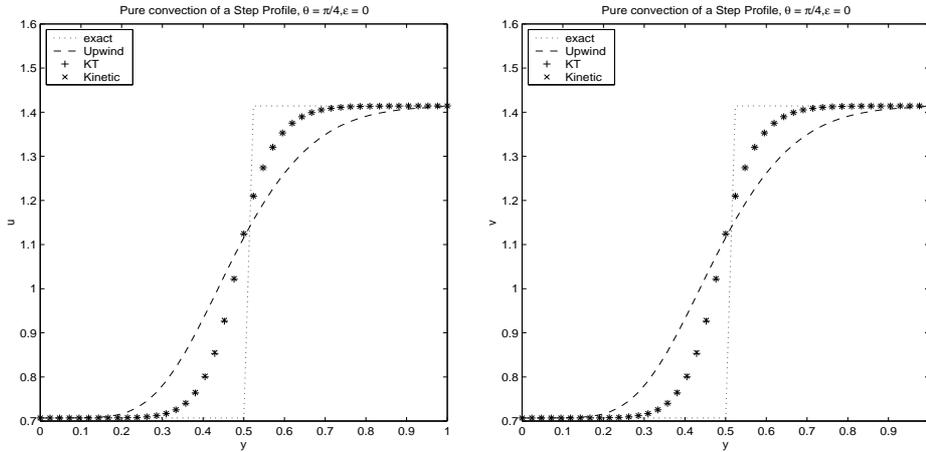
Alternatively the local flow velocity can be used:

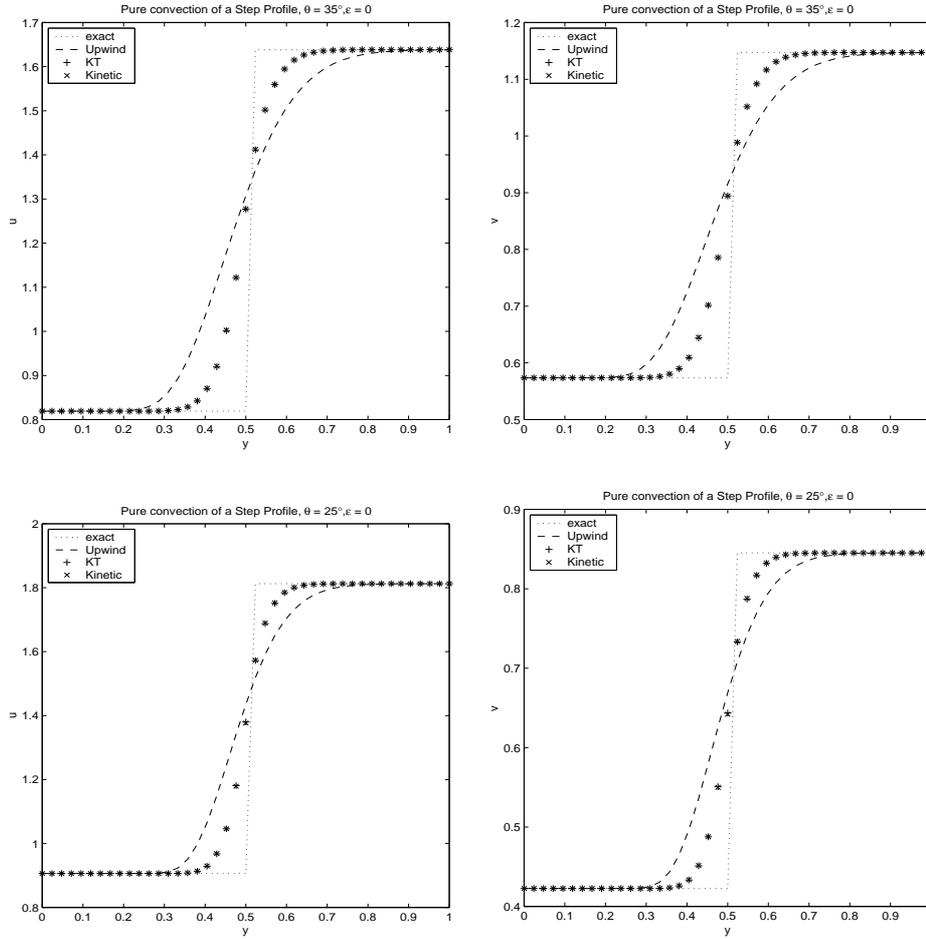
$$c_1(i, j) = \max\{ |2u_{i+1j}|, |2u_{ij}| \}, \quad c_2(i, j) = \max\{ |2v_{ij+1}|, |2v_{ij}| \}$$

In our numerical tests the local flow velocity is used.

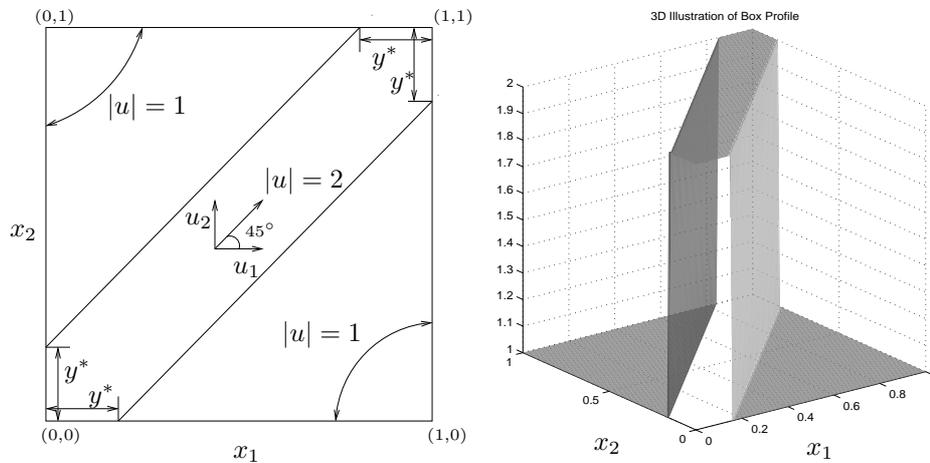
The resulting nonlinear system is solved by a GMRES-based solver described and implemented by Kelley [16].

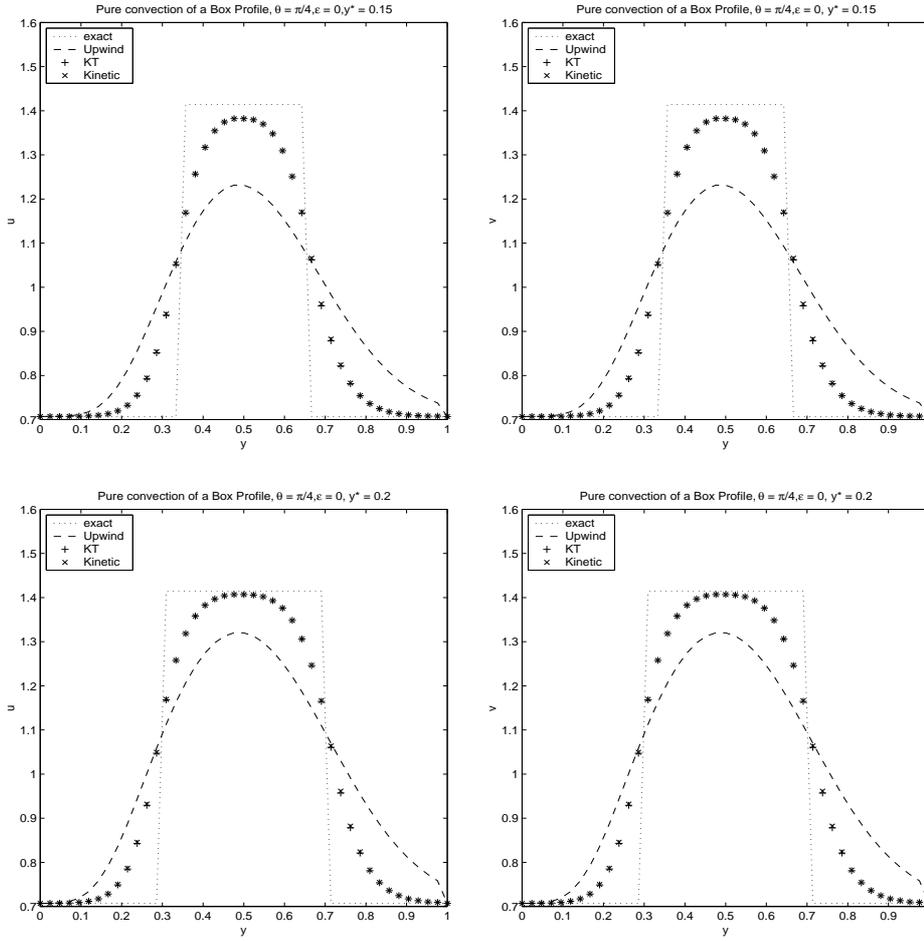
The results are plotted in the following figures. They show the computed profile at the line $x_1 = \frac{1}{2}$ for both the velocity components u_1 and u_2 . In the figures we make a comparison of the first order upwind method, the second order approach by Kurganov and Tadmor [18] and the kinetic second order approach developed here. We always use the local flow velocity to determine c_1 and c_2 .



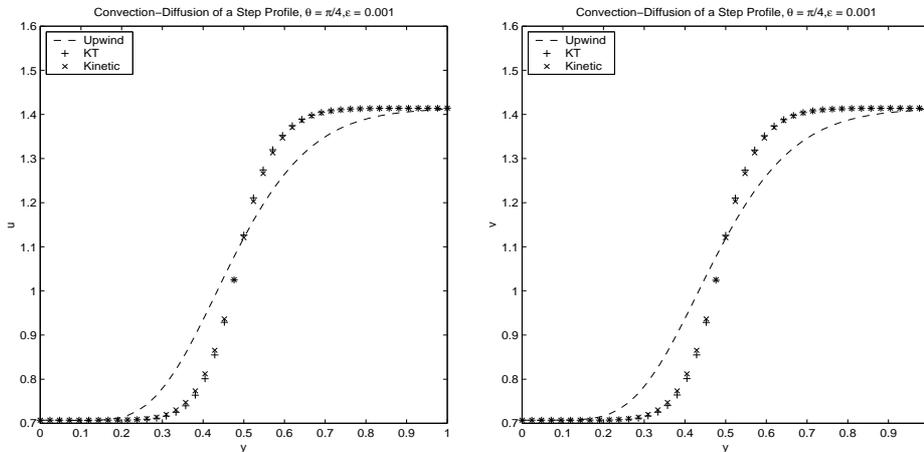


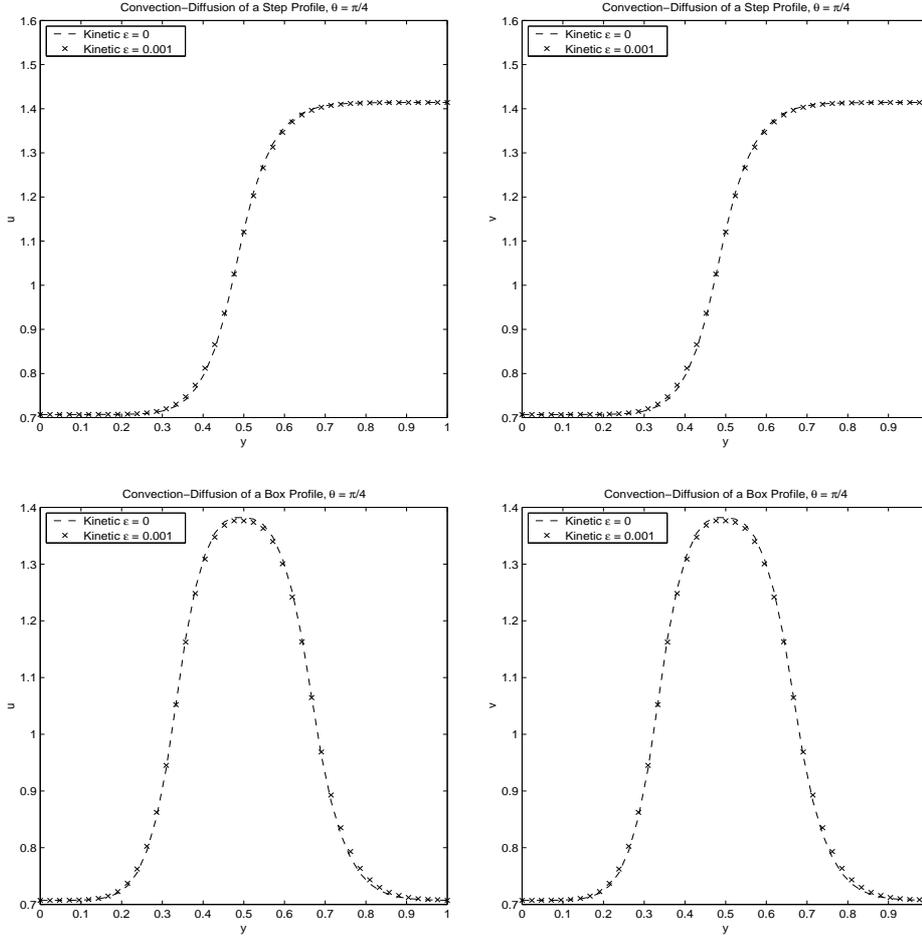
The second example is pure convection of a box-shaped profile:
 We consider a box-shaped profile as shown in the figure below. As in the previous example we compare approximations of the profile across a vertical plane in the middle of the solution domain. We compare the different schemes using a uniform 41×41 mesh.





The figures show that in the cases considered here the results are comparable to those obtained by the Kurganov-Tadmor approach. Further, the convection-diffusion equation for the step-profile is considered. We choose $\epsilon = 0.001$. Just like in the previous examples we took a uniform 41×41 mesh.





Conclusion

Starting from special discretizations of the Boltzmann equation we have obtained corresponding discretizations of the incompressible Navier Stokes equation in the diffusive, low Mach number limit. In particular, a second order slope limiting approach on the kinetic level gives rise to a limiter for the nonlinear term in the Navier Stokes equation which exhibits an interesting structure. From the simple numerical tests one can conclude that the new scheme yields results which are comparable to those obtained by other second order methods. A test of the new limiter in more complicated flow situations with strongly locally varying velocity fields is in preparation.

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