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A NEW PERSPECTIVE ON KINETIC SCHEMES*

MICHAEL JUNK†

Abstract. Compared to standard numerical methods for hyperbolic systems of conservation laws, kinetic schemes model propagation of information by particles instead of waves. In this article, the wave and the particle concept are shown to be closely related. Moreover, a general approach to the construction of kinetic schemes for hyperbolic conservation laws is given that summarizes several approaches discussed by other authors.

Key words. hyperbolic systems of conservation laws, kinetic schemes

AMS subject classifications. 35L65, 35L45, 76M25, 82C40

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1. Introduction.

Kinetic schemes have originally been used to construct approximate solutions of the Euler equations of gas dynamics, but the idea has also been extended to other conservation laws. For scalar equations, the approach is very successful because the main ingredient, a suitable equilibrium distribution that generalizes the Maxwellian velocity distribution function of a gas in local thermodynamical equilibrium, is available. Extensions to systems of conservation laws are also possible but the construction of suitable equilibrium distributions is more difficult.

In this article, our goal is to find a construction principle for equilibrium distributions that applies to general systems of conservation laws and that reduces to the known results in the scalar case. After a brief description of the classical case of Euler equations (section 2), the general framework of kinetic (or particle) formulations is introduced, followed by the definition of a kinetic scheme in section 4. In the case of linear hyperbolic systems the kinetic formulation is shown to be closely related to the wave approach based on Fourier analysis. In fact, it turns out in section 5 that, for a suitable choice of the equilibrium distribution function, the kinetic scheme yields the exact solution of the general linear Cauchy problem. In section 6, an extension of the approach gives rise to a general construction principle for equilibrium distributions.

A consistency and stability analysis singles out a class of hyperbolic systems for which the kinetic approach seems to be extremely well suited. This class contains all linear equations, all scalar conservation laws, and some systems. The members of the class are characterized by the fact that the kinetic scheme has infinite order of consistency and that its linearization is the exact solution of the linearized problem. A hyperbolic system belongs to this class if the Jacobian matrix of the fluxes satisfies certain integrability conditions. In sections 8–10, the construction of equilibrium distributions is applied to several specific examples recovering many approaches proposed by other authors. Also, the relation to the method in [7] is investigated.

2. A particle approach for Euler equations.

To explain how a kinetic (particle) model can be used to approximate solutions of hyperbolic conservation systems, we focus on the important example of Euler equations in gas dynamics (which we

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http://www.siam.org/journals/sinum/38-5/36285.html
†FB Mathematik, Universität Kaiserslautern, Erwin-Schrödinger-Straße, 67663 Kaiserslautern, Germany (junk@mathematik.uni-kl.de).

write using Einstein's summation convention),

$$\begin{aligned} \partial_t \rho + \partial_{x_j} \rho u_j &= 0, \\ \partial_t \rho u_i + \partial_{x_j} \rho u_i u_j + \partial_{x_i} p &= 0, \\ \partial_t \rho\epsilon + \partial_{x_j} (\rho\epsilon + p) u_j &= 0. \end{aligned} \tag{2.1}$$

In this continuum description of a gas, the densities of the conserved quantities mass, momentum, and energy are ρ, $\rho\boldsymbol{u}$, and $\rho\epsilon$. The vector $\boldsymbol{u}$ is the velocity of the gas and p is the pressure. For an ideal gas, the pressure satisfies the relationship $p = \rho T$ (the gas constant is suppressed by choosing an appropriate unit for the temperature T). For simplicity, we consider the case of a monoatomic gas, where the temperature is related to the energy by $T = 2/3(\epsilon - |\boldsymbol{u}|^2/2)$. The idea to solve (2.1) with a particle method has a clear physical origin. Indeed, the continuum description (2.1) can be refined by taking the atomic structure of the gas into account. For the case of rarefied gases this can be done with the theory of Boltzmann equation. In this approach, the basic quantity is a particle distribution function $f_\delta(t, \boldsymbol{x}, \boldsymbol{v})$ that describes the density of gas atoms with velocity $\boldsymbol{v}$ at position $\boldsymbol{x}$ and time t. The gas atoms (i.e., the *particles*) move freely in space unless they undergo collisions. The corresponding evolution of f_δ is given by the Boltzmann equation

$$\partial_t f_\delta + v_j \partial_{x_j} f_\delta = \frac{1}{\delta} Q(f_\delta). \tag{2.2}$$

The left-hand side of (2.2) describes free flow of particles, whereas collisions are modeled by the operator Q. In the kinetic approach, conservation of mass, momentum, and energy is ensured by the property of the collision operator that velocity integrals of $Q(f_\delta)$ weighted with 1, $\boldsymbol{v}$, and $|\boldsymbol{v}|^2$ vanish. The parameter δ in (2.2) denotes the Knudsen number, which is a measure for the mean free path between particle collisions (for details see [10]). A connection between the two descriptions (2.1) and (2.2) is obtained in a limit where particle collisions are dominant ($\delta \to 0$). In this asymptotic case, we find in lowest order $Q(f) = 0$ that implies, due to the structure of the collision operator, that the limit density f has the form of a Maxwellian

$$\mathcal{M}(\rho, \boldsymbol{u}, T; \boldsymbol{v}) = \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{|\boldsymbol{v} - \boldsymbol{u}|^2}{2T} \right). \tag{2.3}$$

This clearly relates the variables f and ρ, $\boldsymbol{u}$, ϵ of both approaches.

Assuming that formally $Q(f_\delta)/\delta \to \bar{Q}$ for $\delta \to 0$, we obtain the limiting evolution

$$\partial_t f + v_j \partial_{x_j} f = \bar{Q} \qquad \text{and} \qquad f = \mathcal{M}. \tag{2.4}$$

Note that $\bar{Q}$ is an additional unknown in the problem but the constraint $f = \mathcal{M}$ also yields an additional equation. Hence $\bar{Q}$ can be viewed as a Lagrange multiplier which ensures the constraint. (For the case of scalar conservation laws, these observations have been made precise in [24].)

To see that the limiting evolution (2.4) implies the Euler evolution (2.1), we multiply (2.4) by 1, $\boldsymbol{v}$, $\frac{1}{2}|\boldsymbol{v}|^2$ and integrate over $\boldsymbol{v}$:

$$\partial_t \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \boldsymbol{v} \\ \frac{1}{2}|\boldsymbol{v}|^2 \end{pmatrix} \mathcal{M}\, d\boldsymbol{v} + \partial_{x_j} \int_{\mathbb{R}^3} v_j \begin{pmatrix} 1 \\ \boldsymbol{v} \\ \frac{1}{2}|\boldsymbol{v}|^2 \end{pmatrix} \mathcal{M}\, d\boldsymbol{v} = \boldsymbol{0}. \tag{2.5}$$

Here we have used the conservation property of the operator Q, which we assume to carry over to $\bar{Q}$ in the limit $\delta \to 0$. Now, using the explicit form (2.3) of $\mathcal{M}$, we calculate

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \boldsymbol{v} \\ \frac{1}{2}|\boldsymbol{v}|^2 \end{pmatrix} \mathcal{M}\, d\boldsymbol{v} = \begin{pmatrix} \rho \\ \rho \boldsymbol{u} \\ \rho\epsilon \end{pmatrix}, \qquad \int_{\mathbb{R}^3} v_j \begin{pmatrix} 1 \\ v_i \\ \frac{1}{2}|\boldsymbol{v}|^2 \end{pmatrix} \mathcal{M}\, d\boldsymbol{v} = \begin{pmatrix} \rho u_j \\ \rho u_i u_j + p\delta_{ij} \\ (\rho\epsilon + p)u_j \end{pmatrix} \tag{2.6}$$

so that, indeed, (2.5) turns into (2.1).

In the following, we call (2.4) a *kinetic formulation* of the Euler equation. Note that the kinetic formulation describes the Euler evolution in terms of a particle ensemble that moves according to the equation $\partial_t f + v_j \partial_{x_j} f = \bar{Q}$ subject to the constraint $f = \mathcal{M}$ on the velocity distribution of the particles. The advantage of the kinetic formulation in comparison to the original system (2.1) is the simpler structure: the differential operator $\partial_t + v_j \partial_{x_j}$ is scalar and linear in contrast to the nonlinear operator in (2.1). In particular, any numerical method known for the simple advection equation can directly be applied to (2.4). A corresponding discretization of the Euler system is obtained by multiplying the discretized version of (2.4) with 1, $\boldsymbol{v}$, and $\frac{1}{2}|\boldsymbol{v}|^2$, integrating over $\boldsymbol{v}$, and observing the constraint and the conservation property of $\bar{Q}$. In [29], the kinetic formulation is discretized in both $\boldsymbol{x}$ and $\boldsymbol{v}$. A similar approach is taken in [28], where a more efficient, semidiscrete form of (2.4) is used. The latter approach was also developed in [13] and extended in several directions by exploiting the possibility to discretize (2.4) with different methods [14, 15, 16].

Taking into account that the relation between the kinetic formulation (2.4) and the Euler system relies only on the property (2.6) of the Maxwellian, it is not surprising that any other function $M(\rho, \boldsymbol{u}, T; \boldsymbol{v})$ that satisfies (2.6) also leads to a kinetic formulation if $f = \mathcal{M}$ in (2.4) is replaced by the constraint $f = M$. This approach has been applied to the Euler system in [22, 30]. Further examples of modified Maxwellians can be found in [26], where M is derived using a maximum entropy principle. A general construction principle of Maxwellians is also given in [3]. Another generalization of the kinetic formulation is obtained by modifying the velocity space. In fact, instead of using $\mathbb{R}^3$ with Lebesgue measure, one can work with more general measure spaces (see [3, 25]).

3. A particle approach for general hyperbolic systems. In the following, we will consider general, autonomous hyperbolic problems of the form

$$\partial_t \boldsymbol{U}(t, \boldsymbol{x}) + \partial_{x_j} \boldsymbol{F}^j(\boldsymbol{U}(t, \boldsymbol{x})) = \boldsymbol{0}, \qquad \boldsymbol{U}(0, \boldsymbol{x}) = \boldsymbol{U}^0(\boldsymbol{x}) \tag{3.1}$$

with $\boldsymbol{x} \in \mathbb{R}^d$. We assume that the unknowns $\boldsymbol{U} = (U_1, \ldots, U_m)^t$ are contained in a connected open set $\mathcal{S} \subset \mathbb{R}^m$ and that $\boldsymbol{F}^j : \mathcal{S} \mapsto \mathbb{R}^m$ are C^1-functions. In the generic case $d > 1$ and $m > 1$, we also assume that $\mathcal{S}$ is simply connected. Note that (3.1) is *hyperbolic* if all linear combinations $\xi_j A^j(\boldsymbol{U})$ of the Jacobian matrices $A^j(\boldsymbol{U}) = \nabla \boldsymbol{F}^j(\boldsymbol{U})$ of the fluxes have only real eigenvalues for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and all $\boldsymbol{U} \in \mathcal{S}$.

Introducing $\langle \cdot, \cdot \rangle_{\boldsymbol{v}}$ to denote integrals over $\boldsymbol{v} \in \mathbb{R}^d$, we call, in accordance with our considerations above,

$$(\partial_t + v_j \partial_{x_j}) \boldsymbol{\Phi} = \boldsymbol{Q}, \qquad \boldsymbol{\Phi}(t, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{\mu}(\boldsymbol{U}(t, \boldsymbol{x}); \boldsymbol{v}), \qquad \langle \boldsymbol{Q}, 1 \rangle_{\boldsymbol{v}} = \boldsymbol{0} \tag{3.2}$$

a *kinetic formulation* of the general system (3.1) if the *constraint function* $\boldsymbol{\mu} : \mathcal{S} \times \mathbb{R}^d \mapsto \mathbb{R}^m$ satisfies the *consistency conditions*

$$\langle \boldsymbol{\mu}(\boldsymbol{U}; \boldsymbol{v}), 1 \rangle_{\boldsymbol{v}} = \boldsymbol{U} \qquad \text{and} \qquad \langle \boldsymbol{\mu}(\boldsymbol{U}; \boldsymbol{v}), v_j \rangle_{\boldsymbol{v}} = \boldsymbol{F}^j(\boldsymbol{U}). \tag{3.3}$$

Again, the kinetic formulation describes the evolution of each U_i in terms of a particle ensemble with velocity distribution $\boldsymbol{v} \mapsto \mu_i(\boldsymbol{U};\boldsymbol{v})$ that moves according to the transport equation $(\partial_t + v_j\partial_{x_j})\Phi_i = Q_i$.

The particular case of Euler equations is recovered by setting

$$\boldsymbol{U} = \begin{pmatrix} \rho \\ \rho\boldsymbol{u} \\ \rho\epsilon \end{pmatrix}, \qquad \boldsymbol{\mu}(\boldsymbol{U};\boldsymbol{v}) = \begin{pmatrix} 1 \\ \boldsymbol{v} \\ \frac{1}{2}|\boldsymbol{v}|^2 \end{pmatrix} \mathcal{M}(\rho,\boldsymbol{u},T;\boldsymbol{v}), \qquad \boldsymbol{Q} = \begin{pmatrix} 1 \\ \boldsymbol{v} \\ \frac{1}{2}|\boldsymbol{v}|^2 \end{pmatrix} \bar{Q}. \tag{3.4}$$

Here the first and last component of $\boldsymbol{\mu}$ are the mass and energy distribution of the particle ensemble and the other components represent the momentum distribution.

For the case of scalar equations ($m = 1$) a constraint function $\boldsymbol{\mu}$ has been derived and investigated by several authors [2, 4, 17, 27]. The same constraint will also follow from our construction principle, which can be viewed as a generalization of the approach in [18] for linear hyperbolic systems in one space dimension. For systems of conservation laws, constraint functions have mainly been proposed in special cases [1, 8, 19, 12], but there are also general approaches to the construction of constraints [3, 18].

4. A kinetic scheme. We have already noted that kinetic formulations can be discretized in many ways, giving rise to different kinetic schemes. Here we focus on the construction of constraint functions $\boldsymbol{\mu}$ and therefore restrict ourselves to a simple *semidiscrete* approximation of (3.2), which has also been used in [26, 22, 2] and which is sometimes called the transport-collapse method [6]. The basic idea is to enforce the constraint $\boldsymbol{\Phi} = \boldsymbol{\mu}$ only at $t_n = n\Delta t$, leading to a purely temporal discretization. Starting with $\boldsymbol{\Phi}(t_n,\boldsymbol{x},\boldsymbol{v}) = \boldsymbol{\mu}(\boldsymbol{U}^n(\boldsymbol{x});\boldsymbol{v})$ we neglect the Lagrange multiplier $\boldsymbol{Q}$ for $t_n < t < t_{n+1}$ and thus have to solve the free flow equation $(\partial_t + v_j\partial_{x_j})\boldsymbol{\Phi} = \boldsymbol{0}$

$$\boldsymbol{\Phi}(t,\boldsymbol{x},\boldsymbol{v}) = \boldsymbol{\mu}(\boldsymbol{U}^n(\boldsymbol{x} - (t-t_n)\boldsymbol{v});\boldsymbol{v}).$$

At the end of the time step, the moment vector $\boldsymbol{U}^{n+1}(\boldsymbol{x}) = \langle \boldsymbol{\Phi}(t_{n+1},\boldsymbol{x},\boldsymbol{v}), 1\rangle_{\boldsymbol{v}}$ is used to reenforce the constraint $\boldsymbol{\Phi} = \boldsymbol{\mu}$. Altogether, we can give an inductive definition of the kinetic scheme which we are going to consider in the following:

> *Let $\boldsymbol{U}^0 : \mathbb{R}^d \mapsto \mathcal{S}$ be the initial value for problem (3.1) and let $t_n = n\Delta t$ for some $\Delta t > 0$ and $n \in \mathbb{N}_0$. If $\boldsymbol{U}^n$ is already constructed and is a function with values in $\mathcal{S}$, we set*
>
> $$\boldsymbol{U}_{\boldsymbol{\mu}}(t,\boldsymbol{x}) = \langle \boldsymbol{\mu}(\boldsymbol{U}^n(\boldsymbol{x} - (t-t_n)\boldsymbol{v});\boldsymbol{v}), 1\rangle_{\boldsymbol{v}}, \qquad t_n \le t \le t_{n+1}$$
>
> *with the value $\boldsymbol{U}^{n+1}(\boldsymbol{x}) = \boldsymbol{U}_{\boldsymbol{\mu}}(t_{n+1},\boldsymbol{x})$ at the end of the time step.*

To check that $\boldsymbol{U}_{\boldsymbol{\mu}}$ is an approximation of the solution of (3.1) we use a Taylor expansion around $t_0 = 0$ which suffices due to the iterative structure of the algorithm. With (3.3) we have

$$\boldsymbol{U}_{\boldsymbol{\mu}}(0,\boldsymbol{x}) = \big\langle \boldsymbol{\mu}(\boldsymbol{U}^0(\boldsymbol{x});\boldsymbol{v}), 1\big\rangle_{\boldsymbol{v}} = \boldsymbol{U}^0(\boldsymbol{x})$$

and

$$\partial_t \boldsymbol{U}_{\boldsymbol{\mu}}(t,\boldsymbol{x})|_{t=0} = \big\langle -v_j\partial_{x_j}\boldsymbol{\mu}(\boldsymbol{U}^0(\boldsymbol{x});\boldsymbol{v}), 1\big\rangle_{\boldsymbol{v}} = -\partial_{x_j}\boldsymbol{F}^j(\boldsymbol{U}^0(\boldsymbol{x}))$$

so that indeed

$$\boldsymbol{U}_{\boldsymbol{\mu}}(t,\boldsymbol{x}) = \boldsymbol{U}^0(\boldsymbol{x}) - \partial_{x_j}\boldsymbol{F}^j(\boldsymbol{U}^0(\boldsymbol{x}))\Delta t + \mathcal{O}(\Delta t^2).$$

In other words, the consistency conditions (3.3) guarantee that $\boldsymbol{U}_{\boldsymbol{\mu}}$ is at least a first-order consistent approximation to the solution of (3.1). The reason why we cannot expect a higher consistency order in general is due to the neglect of the source term $\boldsymbol{Q}$. In fact, whenever $\boldsymbol{Q}$ vanishes only initially, the free-flow equation is just a first-order consistent approximation to (3.2) which carries over to the hyperbolic system by taking moments.

For the construction of $\boldsymbol{\mu}$, it is important to note that the consistency conditions (3.3) do not determine $\boldsymbol{\mu}$ uniquely. Hence there should be a selection mechanism to single out an appropriate constraint function. One possibility is to select a constraint function which is optimal with respect to a convex functional (entropy) while satisfying the consistency conditions. As a by-product, the resulting kinetic scheme also satisfies an entropy inequality (see [26]). In this work, we pursue a different optimality condition: we select a constraint function $\mathring{\boldsymbol{\mu}}$ which maximizes the order of consistency of the semidiscrete kinetic scheme introduced above. This optimality result holds in the class of constraint functions $\boldsymbol{U} \mapsto \boldsymbol{\mu}(\boldsymbol{U})$ for which each component μ_i is contained in $\mathcal{K}$, the set of continuous mappings from $\mathcal{S}$ into the space $\mathcal{E}'(\mathbb{R}^d)$ of compactly supported distributions which satisfy an additional locally uniform estimate (see Definition A.4 in the appendix for details). For each $\boldsymbol{\mu} \in \mathcal{K}^m$, the corresponding kinetic scheme gives rise to the approximation

$$\boldsymbol{U}_{\boldsymbol{\mu}}(t, \boldsymbol{x}) := \left\langle \boldsymbol{\mu}(\boldsymbol{U}^0(\boldsymbol{x} - \boldsymbol{v}t); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}},$$

where $\boldsymbol{U}^0$ is taken from the set of initial values

$$\mathcal{J} := \left\{ \boldsymbol{V} : \mathbb{R}^d \mapsto \mathcal{S} : \boldsymbol{V} \in C^\infty(\mathbb{R}^d)^m,\ \boldsymbol{V}(\mathbb{R}^d) \text{ compact in } \mathcal{S} \right\}.$$

If $\boldsymbol{U}(t, \boldsymbol{x})$ is the corresponding classical solution of (3.1) on $(-T, T) \times \mathbb{R}^d$ for some $T > 0$, we define the consistency as

$$\operatorname{con}(\boldsymbol{\mu}) := \inf \left\{ n - 1 : n \in \mathbb{N}_0,\ \partial_t^n \boldsymbol{U}|_{t=0} \neq \partial_t^n \boldsymbol{U}_{\boldsymbol{\mu}}|_{t=0} \text{ for some } \boldsymbol{U}^0 \in \mathcal{J} \right\}.$$

With this notation, the optimality of the constraint function $\mathring{\boldsymbol{\mu}}$ constructed below can be formulated as $\operatorname{con}(\mathring{\boldsymbol{\mu}}) \geq \operatorname{con}(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathcal{K}^m$. It turns out that the maximal order of consistency obtainable with kinetic schemes of the above type depends on structural properties of the hyperbolic system related to integrability properties of products of the Jacobian matrices $A^j(\boldsymbol{u}) = \nabla \boldsymbol{F}^j(\boldsymbol{u})$. Before stating the result in the following theorem, we note that a matrix valued function $B : \mathcal{S} \mapsto \mathbb{R}^{m \times d}$ is called *exact* if $B = \nabla \boldsymbol{b}$ for some $\boldsymbol{b} : \mathcal{S} \mapsto \mathbb{R}^m$, or in other words, if B has a primitive.

THEOREM 4.1. *An explicit expression for the optimal consistency* $\operatorname{con}(\mathring{\boldsymbol{\mu}})$ *is*

$$\operatorname{con}(\mathring{\boldsymbol{\mu}}) = \sup \left\{ n \in \mathbb{N} : (\xi_j A^j)^{\bar{n}-1} A^k \text{ is exact for all } \bar{n} \leq n,\ \boldsymbol{\xi} \in \mathbb{R}^d,\ k = 1, \ldots, d \right\}.$$

For systems in one space dimension ($d = 1$)*, the condition reduces to exactness of* $\bar{n}$*-fold products of the Jacobian* A^1 *for all* $\bar{n} \leq n$*. In the scalar case* ($m = 1$) *and for linear systems, the maximal consistency order is always infinite. Since* $\operatorname{con}(\mathring{\boldsymbol{\mu}}) \geq 1$ *for any system, the kinetic scheme based on* $\mathring{\boldsymbol{\mu}}$ *is always consistent. If* $\operatorname{con}(\boldsymbol{\mu}) = \infty$ *for some* $\boldsymbol{\mu} \in \mathcal{K}^m$*, then* $\boldsymbol{\mu}$ *is essentially given by* $\mathring{\boldsymbol{\mu}}$*, i.e.,* $\boldsymbol{\mu} = \mathring{\boldsymbol{\mu}}_\eta + \boldsymbol{C}$*, where* $\boldsymbol{C} \in [\mathcal{E}'(\mathbb{R}^d)]^m$ *is independent of* $\boldsymbol{U}$ *and satisfies* $\langle \boldsymbol{C}, 1 \rangle_{\boldsymbol{v}} = \boldsymbol{0}$.

Proofs for the results in this theorem can be found in [20, 21]. We remark that the infinite order of consistency in the case of linear systems follows from the fact

that constant matrices can always be viewed as Jacobians of linear functions and thus are exact. For scalar equations, we note that $(\xi_j A^j)^{\bar{n}-1}A^k$ is a continuous, scalar function in the scalar state variable U. Using the fundamental theorem of calculus, a primitive can be obtained simply by integration with variable upper bound which shows exactness for any $\boldsymbol{\xi}, \bar{n}$ and k. Already for systems in one space dimension, the situation is very different. Even $(A^1)^2 = \nabla \boldsymbol{F}^1 \nabla \boldsymbol{F}^1$ need not be exact so that the kinetic scheme will yield only a first-order consistent method in general. In such cases, the optimality of $\mathring{\boldsymbol{\mu}}$ is only a weak statement because any constraint with $\mathrm{con}(\boldsymbol{\mu}) = 1$ will also be optimal. Nevertheless, $\mathring{\boldsymbol{\mu}}$ is singled out by the fact that its construction principle automatically yields the unique (up to constants) optimal solution in the case $\mathrm{con}(\mathring{\boldsymbol{\mu}}) = \infty$. Some examples of such cases will be mentioned in sections 9 and 10.

We conclude with the remark that other discretizations of (3.2) can, of course, lead to kinetic schemes with higher orders of consistency if the term $\boldsymbol{Q}$ is not completely neglected. The semidiscrete approximation chosen here, however, is very well suited for the construction of constraint functions which is our main objective.

5. Equivalence of wave and particle approach. To illustrate the similarities between the particle approach based on a kinetic formulation and the more common wave approach, let us briefly recollect the case of a *linear* hyperbolic system, where the flux functions are of the form $\boldsymbol{F}^j(\boldsymbol{U}) = A^j \boldsymbol{U}$ with constant matrices $A^j \in \mathbb{R}^{m\times m}$:

$$\partial_t \boldsymbol{U}(t,\boldsymbol{x}) + A^j \partial_{x_j} \boldsymbol{U}(t,\boldsymbol{x}) = \boldsymbol{0}, \qquad \boldsymbol{U}(0,\boldsymbol{x}) = \boldsymbol{U}^0(\boldsymbol{x}). \tag{5.1}$$

Applying the Fourier transform in the space variable, we obtain a system of ordinary differential equations

$$\partial_t \hat{\boldsymbol{U}}(t,\boldsymbol{\xi}) + i\xi_j A^j \hat{\boldsymbol{U}}(t,\boldsymbol{\xi}) = \boldsymbol{0}, \qquad \hat{\boldsymbol{U}}(0,\boldsymbol{\xi}) = \hat{\boldsymbol{U}}^0(\boldsymbol{\xi})$$

(with i being the imaginary unit) which has the solution

$$\hat{\boldsymbol{U}}(t,\boldsymbol{\xi}) = \hat{E}_t(\boldsymbol{\xi})\hat{\boldsymbol{U}}^0(\boldsymbol{\xi}), \qquad \hat{E}_t(\boldsymbol{\xi}) = \exp(-it\xi_j A^j). \tag{5.2}$$

Transforming back, we can write the solution as a superposition of plane waves

$$\boldsymbol{U}(t,\boldsymbol{x}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \boldsymbol{W}_{\boldsymbol{\xi}}(t,\boldsymbol{x})\, d\boldsymbol{\xi}, \qquad \boldsymbol{W}_{\boldsymbol{\xi}}(t,\boldsymbol{x}) = \exp\big(i(\boldsymbol{\xi}\cdot\boldsymbol{x})\,\mathrm{I} - it\xi_j A^j)\big)\hat{\boldsymbol{U}}^0(\boldsymbol{\xi}).$$

The reformulation of this wave approach into a particle formulation simply relies on the property of the Fourier transform to convert products into convolutions. To see this, we use relation (5.2), denoting the Fourier transform by $\mathcal{F}$:

$$\boldsymbol{U}(t,\boldsymbol{x}) = \mathcal{F}^{-1}_{\boldsymbol{\xi}}\left(\hat{E}_t(\boldsymbol{\xi})\hat{\boldsymbol{U}}^0(\boldsymbol{\xi})\right)\Big|_{\boldsymbol{x}}.$$

With the abbreviation $E_t := \mathcal{F}^{-1}\hat{E}_t$ we obtain formally $\boldsymbol{U}(t,\boldsymbol{x}) = E_t * \boldsymbol{U}^0(\boldsymbol{x})$, or more explicitly

$$\boldsymbol{U}(t,\boldsymbol{x}) = \big\langle E_t(\boldsymbol{y})\boldsymbol{U}^0(\boldsymbol{x}-\boldsymbol{y}), 1\big\rangle_{\boldsymbol{y}}. \tag{5.3}$$

(To avoid technicalities at this point we proceed purely formally. Actually, E_t has to be interpreted in the sense of distributions which we do later.) Using the fact that

$$\mathcal{F}^{-1}_{\boldsymbol{\xi}}\phi(t\boldsymbol{\xi})\Big|_{t\boldsymbol{v}} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\phi(t\boldsymbol{\xi})\exp(i\boldsymbol{\xi}\cdot t\boldsymbol{v})\, d\boldsymbol{\xi} = \frac{1}{t^d}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\phi(\boldsymbol{\eta})\exp(i\boldsymbol{\eta}\cdot\boldsymbol{v})\, d\boldsymbol{\eta}$$

for any test function ϕ, we find accordingly $E_t(\boldsymbol{v}t) = E_1(\boldsymbol{v})/t^d$. Hence the change of variables $\boldsymbol{y} = t\boldsymbol{v}$ in (5.3) yields

$$\boldsymbol{U}(t,\boldsymbol{x}) = \left\langle E_1(\boldsymbol{v})\boldsymbol{U}^0(\boldsymbol{x}-t\boldsymbol{v}),1\right\rangle_{\boldsymbol{v}}. \tag{5.4}$$

This result suggests to introduce the vector constraint function

$$\mathring{\boldsymbol{\mu}}_{lin}(\boldsymbol{U};\boldsymbol{v}) := E(\boldsymbol{v})\boldsymbol{U}, \qquad E = E_1 = \mathcal{F}_{\boldsymbol{\xi}}^{-1}\exp(-i\xi_j A^j). \tag{5.5}$$

Then the solution (5.4) of the linear hyperbolic system coincides with the approximation $\boldsymbol{U}_{\mathring{\boldsymbol{\mu}}_{lin}}$ obtained with the kinetic scheme defined in section 4

$$\boldsymbol{U}_{\mathring{\boldsymbol{\mu}}_{lin}}(t,\boldsymbol{x}) = \left\langle \mathring{\boldsymbol{\mu}}_{lin}(\boldsymbol{U}^0(\boldsymbol{x}-t\boldsymbol{v});\boldsymbol{v}),1\right\rangle_{\boldsymbol{v}}.$$

This remarkable result is partly related to our choice of the kinetic scheme, but it demonstrates that the concept of kinetic formulations is intimately related to hyperbolic equations. In particular, the derivation shows that solving the free transport equation together with velocity averaging is closely connected to convolution.

Since the kinetic scheme based on $\mathring{\boldsymbol{\mu}}_{lin}$ yields the exact solution, it is evident that the consistency conditions (3.3) are satisfied. This property can also be checked directly. Translating $\boldsymbol{v}$-moments of E into $\boldsymbol{\xi}$-derivatives of the Fourier transform at $\boldsymbol{\xi} = \boldsymbol{0}$, we have

$$\langle E,1\rangle_{\boldsymbol{v}} = \mathcal{F}(E)(\boldsymbol{0}) = \exp(-i\xi_j A^j)\big|_{\boldsymbol{\xi}=\boldsymbol{0}} = \mathrm{I} \tag{5.6}$$

and

$$\langle E,v_k\rangle_{\boldsymbol{v}} = \mathcal{F}(v_k E)(\boldsymbol{0}) = i\partial_{\xi_k}\exp(-i\xi_j A^j)\big|_{\boldsymbol{\xi}=\boldsymbol{0}} = A^k \tag{5.7}$$

so that

$$\langle \mathring{\boldsymbol{\mu}}_{lin}(\boldsymbol{U}),1\rangle_{\boldsymbol{v}} = \langle E,1\rangle_{\boldsymbol{v}}\boldsymbol{U} = \boldsymbol{U}, \qquad \langle \mathring{\boldsymbol{\mu}}_{lin}(\boldsymbol{U}),v_k\rangle_{\boldsymbol{v}} = \langle E,v_k\rangle_{\boldsymbol{v}}\boldsymbol{U} = A^k\boldsymbol{U}.$$

6. General constraint functions. For general linear hyperbolic problems like (5.1) we have found an optimal constraint function

$$\mathring{\boldsymbol{\mu}}_{lin}(\boldsymbol{U};\boldsymbol{v}) := E(\boldsymbol{v})\boldsymbol{U}, \qquad E = \mathcal{F}_{\boldsymbol{\xi}}^{-1}\exp(-i\xi_j A^j) \tag{6.1}$$

for which the corresponding kinetic scheme even yields the exact solution. In [18], the same constraint has been derived for the linear case in a single space dimension ($d = 1$) if A^1 has a complete set of eigenvectors. The authors are able to extend the construction of $\boldsymbol{\mu}$ to nonlinear systems if the flux can be put in the form

$$\boldsymbol{F}^1(\boldsymbol{U}) = A^1(\boldsymbol{U})\boldsymbol{U}$$

with $A^1(\boldsymbol{U})$ having pointwise the same properties as required for the linear case (such a representation is possible if the system admits a convex entropy). With this extension, however, the optimality in the linear case (as far as recovering exact solutions is concerned) does not carry over to nonlinear problems. We therefore propose a different generalization of (6.1). First, we rewrite (6.1) as integral over the matrix E which is constant with respect to $\boldsymbol{U}$,

$$\mathring{\boldsymbol{\mu}}_{lin}(\boldsymbol{U}) = \int_{\boldsymbol{0}}^{\boldsymbol{U}} E := \int_0^1 E\,\dot{\boldsymbol{\gamma}}(s)\,ds, \tag{6.2}$$

where $\boldsymbol{\gamma} : [0,1] \mapsto \mathbb{R}^m$ is a curve in the state space $\mathcal{S} = \mathbb{R}^m$ which connects the origin with $\boldsymbol{U}$. To obtain an expression similar to (6.2) in the case of general systems, we assume in the following that $\mathbf{0} \in \mathcal{S}$ and $\boldsymbol{F}^j(\mathbf{0}) = \mathbf{0}$ for $j = 1, \ldots, d$. This can always be achieved by selecting some point $\tilde{\boldsymbol{U}} \in \mathcal{S}$ and going over to the fluxes $\tilde{\boldsymbol{F}}^j(\boldsymbol{V}) = \boldsymbol{F}^j(\boldsymbol{V} + \tilde{\boldsymbol{U}}) - \boldsymbol{F}^j(\tilde{\boldsymbol{U}})$ defined on $\tilde{\mathcal{S}} = \mathcal{S} - \tilde{\boldsymbol{U}}$ which certainly contains $\mathbf{0}$.

A straightforward generalization of (6.1) and (6.2) to the nonlinear case is obtained if we replace the constant matrices A^j by the flux derivatives $A^j(\boldsymbol{U}) = \nabla \boldsymbol{F}^j(\boldsymbol{U})$. Obviously, the matrix E then depends on $\boldsymbol{U} \in \mathcal{S} \subset \mathbb{R}^m$

$$E(\boldsymbol{U}) = \mathcal{F}_{\boldsymbol{\xi}}^{-1} \exp(-i\xi_j A^j(\boldsymbol{U}))$$

so that the line integral of E in (6.2) is no longer trivial. Since we cannot expect that the line integrals are independent of the chosen curves in the state space $\mathcal{S}$, we have to fix properties of the parameterization. If $\boldsymbol{\gamma}(\boldsymbol{U}; s)$ is the parameterization of a curve connecting the origin with $\boldsymbol{U}$ in $\mathcal{S}$, we require that the $\boldsymbol{U}$-dependence of $\boldsymbol{\gamma}$ is reasonably nice and that the integrals of E along the curves are well defined (we speak of $\boldsymbol{F}$*-admissible curves*—for details see Definition A.3 in the appendix). If $\Gamma_{\boldsymbol{U}}$ is the graph of the parameterization $s \mapsto \boldsymbol{\gamma}(\boldsymbol{U}; s)$, we define the line integral of the matrix E as

$$\int_{\Gamma_{\boldsymbol{U}}} E := \int_0^1 E(\boldsymbol{\gamma}(\boldsymbol{U}; s)) \dot{\boldsymbol{\gamma}}(\boldsymbol{U}; s)\, ds,$$

where $\dot{\boldsymbol{\gamma}}$ refers to the s-derivative of $\boldsymbol{\gamma}$. Finally, the proposed generalization of (6.2) to the case of nonlinear hyperbolic systems is given by

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}) := \int_{\Gamma_{\boldsymbol{U}}} E, \qquad E(\boldsymbol{U}) = \mathcal{F}_{\boldsymbol{\xi}}^{-1} \exp(-i\xi_j A^j(\boldsymbol{U})). \tag{6.3}$$

A rigorous description of the mathematical properties of $\mathring{\boldsymbol{\mu}}$ is given in the appendix, where the approach is generalized to entropy conservation laws related to the hyperbolic system.

In section 4, we have already discussed the optimality property of $\mathring{\boldsymbol{\mu}}$ with respect to the order of consistency. However, consistency alone does not fully describe the behavior of the scheme. The second important concept besides consistency is stability. Therefore, in the following section, we consider linear stability of the kinetic scheme and mention the approach of modified equations. In the remaining sections, we then give examples where the Fourier transform in (6.3) is calculated explicitly.

7. Some remarks on stability.

7.1. Linear stability. In general, the constraint function $\boldsymbol{U} \mapsto \mathring{\boldsymbol{\mu}}(\boldsymbol{U})$ depends nonlinearly on $\boldsymbol{U}$ so that the kinetic scheme based on $\mathring{\boldsymbol{\mu}}$ is also nonlinear. However, if the initial value varies only slightly around some value $\bar{\boldsymbol{U}} \in \mathcal{S}$, one can linearize the kinetic scheme and study the stability properties of the resulting scheme which approximates the linearization of (3.1),

$$\partial_t \boldsymbol{W} + A^j(\bar{\boldsymbol{U}}) \partial_{x_j} \boldsymbol{W} = \mathbf{0}, \qquad \boldsymbol{W}|_{t=0} = \boldsymbol{W}^0 = \boldsymbol{U}^0 - \bar{\boldsymbol{U}}. \tag{7.1}$$

A corresponding linearization of the kinetic scheme relies on

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}; \boldsymbol{v}) \approx \mathring{\boldsymbol{\mu}}(\bar{\boldsymbol{U}}; \boldsymbol{v}) + \nabla \mathring{\boldsymbol{\mu}}(\bar{\boldsymbol{U}}; \boldsymbol{v}) \boldsymbol{W}, \qquad \boldsymbol{W} = \boldsymbol{U} - \bar{\boldsymbol{U}},$$

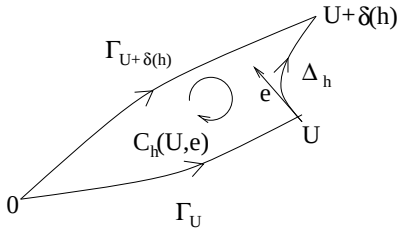

FIG. 7.1. *Closed curve in state space.*

where the gradient of $\mathring{\boldsymbol{\mu}}$ is taken with respect to the state variable $\boldsymbol{U}$. To calculate the derivative of $\mathring{\boldsymbol{\mu}}$ into some direction $\boldsymbol{e} \in \mathbb{R}^m$, we select a curve $\boldsymbol{\delta} \in C^1([0,1], \mathbb{R}^m)$ which satisfies $\boldsymbol{\delta}(0) = \mathbf{0}$ and $\dot{\boldsymbol{\delta}}(0) = \boldsymbol{e}$. Then

$$\nabla \mathring{\boldsymbol{\mu}}(\boldsymbol{U})\boldsymbol{e} = \lim_{h \to 0} \frac{1}{h} \left(\int_{\Gamma_{\boldsymbol{U}+\boldsymbol{\delta}(h)}} E - \int_{\Gamma_{\boldsymbol{U}}} E \right). \tag{7.2}$$

Introducing the graph $\Delta_h(\boldsymbol{U}, \boldsymbol{e}) = \boldsymbol{U} + \boldsymbol{\delta}([0,h])$, we set up the oriented closed curve $C_h(\boldsymbol{U}, \boldsymbol{e}) := \Gamma_{\boldsymbol{U}+\boldsymbol{\delta}(h)} - \Delta_h(\boldsymbol{U}, \boldsymbol{e}) - \Gamma_{\boldsymbol{U}}$ (see Figure 7.1). Hence (7.2) transforms into

$$\nabla \mathring{\boldsymbol{\mu}}(\boldsymbol{U})\boldsymbol{e} = \lim_{h \to 0} \frac{1}{h} \int_{\Delta_h(\boldsymbol{U},\boldsymbol{e})} E + \lim_{h \to 0} \frac{1}{h} \oint_{C_h(\boldsymbol{U},\boldsymbol{e})} E.$$

Using the mean value theorem, the first integral reduces to $E(\boldsymbol{U})\boldsymbol{e}$ and the second one can be rewritten as

$$q(\boldsymbol{U})\boldsymbol{e} = \frac{d}{dh} \oint_{C_h(\boldsymbol{U},\boldsymbol{e})} E \Big|_{h=0}. \tag{7.3}$$

Altogether, we find $\nabla \mathring{\boldsymbol{\mu}}(\boldsymbol{U}) = E(\boldsymbol{U}) + q(\boldsymbol{U})$. In the case $\boldsymbol{U}^0 = \bar{\boldsymbol{U}} + \boldsymbol{W}^0$ with small $\boldsymbol{W}^0$, it is thus reasonable to consider the following approximation of the kinetic scheme:

$$\begin{aligned} \bar{\boldsymbol{U}} + \tilde{\boldsymbol{W}}(t, \boldsymbol{x}) &= \left\langle \mathring{\boldsymbol{\mu}}(\boldsymbol{U}^0(\boldsymbol{x} - t\boldsymbol{v}); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}} \\ &\approx \left\langle \mathring{\boldsymbol{\mu}}(\bar{\boldsymbol{U}}; \boldsymbol{v}) + (E(\bar{\boldsymbol{U}}; v) + q(\bar{\boldsymbol{U}}; v))\boldsymbol{W}^0(\boldsymbol{x} - t\boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}. \end{aligned} \tag{7.4}$$

According to our investigations in section 5, $E(\bar{\boldsymbol{U}}; v)\boldsymbol{W}$ is the constraint function $\mathring{\boldsymbol{\mu}}_{lin}(\boldsymbol{W}; \boldsymbol{v})$ for the linearized system (7.1). For the remainder term in (7.4) we introduce $\boldsymbol{\mu}_q(\boldsymbol{W}; \boldsymbol{v}) = q(\bar{\boldsymbol{U}}; v)\boldsymbol{W}$ so that the linearized kinetic scheme is clearly seen to be the superposition of two linear kinetic schemes

$$\tilde{\boldsymbol{W}}(t, \boldsymbol{x}) \approx \left\langle \mathring{\boldsymbol{\mu}}_{lin}(\boldsymbol{W}^0(\boldsymbol{x} - t\boldsymbol{v}); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}} + \left\langle \boldsymbol{\mu}_q(\boldsymbol{W}^0(\boldsymbol{x} - t\boldsymbol{v}); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}.$$

While the first contribution is the exact solution of the linear system (7.1) (and thus stable), the second one strongly depends on the selected curves and the interplay of these curves with the matrix E which contains the information about the hyperbolic system. Stability problems of the scheme can originate only in the term related to the matrix q defined in (7.3).

At this point it becomes obvious that the kinetic scheme is particularly well suited for those hyperbolic problems for which closed curve integrals over E vanish. In this

case, q is identically zero and the linearized kinetic scheme is the optimal kinetic scheme for the linearized equation.

From the theory of differential one-forms, it is known that vanishing closed curve integrals are related to exactness. Applied to $E = \mathcal{F}_{\boldsymbol{\xi}}^{-1} \exp(-i\xi_j A^j)$, the condition reduces to the requirement that the mapping

$$\boldsymbol{U} \mapsto \exp(-i\xi_j A^j(\boldsymbol{U})) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (\xi_j A^j(\boldsymbol{U}))^n$$

must be exact (i.e., possess a primitive). It can easily be shown [20] that this is just another way of requiring the exactness of all products $\boldsymbol{U} \mapsto (\xi_j A^j(\boldsymbol{U}))^n$. Note that this observation is related to the results of Theorem 4.1. In particular, for all scalar conservation laws, the proposed constraint function yields a linearly stable scheme.

In case the exactness of all products $(\xi_j A^j)^n$ is not given, the additional term related to q does not vanish and instabilities can occur if the Fourier transform of q has amplifying modes.

7.2. The modified equation approach. In general, the kinetic scheme based on $\mathring{\boldsymbol{\mu}}$ yields a first-order accurate solution to problem (3.1) However, by a simple Taylor expansion argument, one can check that the approximation obtained with the kinetic scheme is at least second-order accurate to the so-called *modified equation*

$$\partial_t \boldsymbol{U} + \partial_{x_j} \boldsymbol{F}^j(\boldsymbol{U}) = \frac{1}{2} \Delta t \partial_{x_j} \left[\left(\nabla \left\langle \mathring{\boldsymbol{\mu}}(\boldsymbol{U}; v), v_j v_i \right\rangle_{\boldsymbol{v}} - A^j(\boldsymbol{U}) A^i(\boldsymbol{U}) \right) \partial_{x_i} \boldsymbol{U} \right]. \tag{7.5}$$

The modified equation (7.5) is a nonlinear advection diffusion equation with diffusion coefficients $Q_{ji} = \frac{1}{2}\Delta t \left(\nabla \left\langle \mathring{\boldsymbol{\mu}}, v_j v_i \right\rangle_v - A^j A^i \right)$. Analogous to the considerations in [3], we introduce for fixed $\boldsymbol{U} \in \mathcal{S}$ the linear mapping $Q(\boldsymbol{U}) : (\mathbb{R}^m)^d \mapsto (\mathbb{R}^m)^d$ according to

$$(Q(\boldsymbol{U})\boldsymbol{v})_j := \sum_{i=1}^{d} Q_{ji}(\boldsymbol{U}) \boldsymbol{v}_i, \qquad \boldsymbol{v}_i \in \mathbb{R}^m, \quad i, j = 1, \ldots, d.$$

If Q has negative eigenvalues, we expect (7.5) to behave like the ill posed backward heat equation which roughens the solution during the evolution. Since the kinetic scheme yields a good approximation to (7.5), a similar behavior is then expected for the kinetic scheme. This heuristic argument is the motivation for the following definition of stability.

DEFINITION 7.1. *The kinetic scheme based on $\mathring{\boldsymbol{\mu}}$ is called stable (in the sense of modified equation) if the linear mapping $Q(\boldsymbol{U})$ has only nonnegative eigenvalues for all $\boldsymbol{U} \in \mathcal{S}$.*

We will check this condition in one-dimensional cases where $Q(\boldsymbol{U})$ is an $m \times m$ matrix. As in (5.6), (5.7), one can show for $d = 1$ that $\langle E, v^2 \rangle = (A^1)^2$. Using the relation $\nabla \mathring{\boldsymbol{\mu}}(\boldsymbol{U}) = E(\boldsymbol{U}) + q(\boldsymbol{U})$, we thus conclude for $d = 1$

$$Q(\boldsymbol{U})\boldsymbol{e} = \left\langle q(\boldsymbol{U}; v)\boldsymbol{e}, v^2 \right\rangle_v = \frac{d}{dh} \oint_{C_h(\boldsymbol{U}, \boldsymbol{e})} (A)^2 \Big|_{h=0}.$$

In this formulation we see that $Q(\boldsymbol{U})$ can be interpreted as a measure of nonexactness of $(A^1)^2$ with respect to the family of curves $\{\Gamma_{\boldsymbol{U}}\}$.

8. Scalar equations. In the case of a single conservation law ($m = 1$), the fluxes F^j are scalar functions defined on an interval $\mathcal{S} \subset \mathbb{R}$ and we can combine them in a vector

$$\boldsymbol{F}(U) = (F^1(U), \dots, F^d(U))^t$$

(note that $\boldsymbol{U}$ and the flux vectors $\boldsymbol{F}^j$ are replaced by nonbold symbols U and F^j since they are scalars in this section). Obviously, the U-derivative $\boldsymbol{F}'$ of $\boldsymbol{F}$ is then given by $\boldsymbol{F}' = (A^1, \dots, A^d)$ and thus $\xi_j A^j = \boldsymbol{\xi} \cdot \boldsymbol{F}'$. Since the inverse Fourier transform of $\exp(-i\boldsymbol{\xi} \cdot \boldsymbol{F}')$ is just the delta distribution shifted to $\boldsymbol{F}'$, the constraint function (6.3) turns into

$$\mathring{\mu}(U; \boldsymbol{v}) = \int_0^U \delta(\boldsymbol{v} - \boldsymbol{F}'(s))\, ds = \int_{\mathbb{R}} \mathcal{X}_{[0,U]}(s) \delta(\boldsymbol{v} - \boldsymbol{F}'(s))\, ds, \tag{8.1}$$

where $\mathcal{X}_{[a,b]}$ denotes the indicator function of the interval $[a, b]$ together with the convention that $\mathcal{X}_{[b,a]} = -\mathcal{X}_{[a,b]}$ for $a < b$.

In general, the constraint function (8.1) cannot be simplified much further. However, it is possible to simplify the kinetic scheme based on $\mathring{\mu}$. Setting

$$U^{\circ}_{\mathring{\mu}}(t, \boldsymbol{x}) = \left\langle \mathring{\mu}(U^0(\boldsymbol{x} - \boldsymbol{v}t); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}$$

we obtain for any test function $\psi \in \mathcal{D}(\mathbb{R}^d)$

$$\left\langle U^{\circ}_{\mathring{\mu}}(t, \boldsymbol{x}), \psi(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}} = \left\langle \mathring{\mu}(U^0(\boldsymbol{x}); \boldsymbol{v}), \psi(\boldsymbol{x} + \boldsymbol{v}t) \right\rangle_{(\boldsymbol{x}, \boldsymbol{v})}$$

and, using the structure of $\mathring{\mu}$ in (8.1), we can evaluate the $\boldsymbol{v}$-part of the dual pairing

$$\left\langle U^{\circ}_{\mathring{\mu}}(t, \boldsymbol{x}), \psi(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}} = \int_{\mathbb{R}} \left\langle \mathcal{X}_{[0, U^0(\boldsymbol{x})]}(s), \psi(\boldsymbol{x} + \boldsymbol{F}'(s)t) \right\rangle_{\boldsymbol{x}} ds.$$

Going over to the shifted $\boldsymbol{x}$ variable $\boldsymbol{x} \mapsto \boldsymbol{x} - \boldsymbol{F}'(s)t$, we finally obtain

$$\left\langle U^{\circ}_{\mathring{\mu}}(t, \boldsymbol{x}), \psi(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}} = \left\langle \int_{\mathbb{R}} \mathcal{X}_{[0, U^0(\boldsymbol{x} - \boldsymbol{F}'(s)t)]}(s)\, ds, \psi(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}}.$$

In other words, the approximative solution $U^{\circ}_{\mathring{\mu}}$ of the entropy conservation law can be written as $U^{\circ}_{\mathring{\mu}}(t, \boldsymbol{x}) = \langle g(\boldsymbol{x}, s, t), 1 \rangle_s$, where $g(\boldsymbol{x}, s, t) = \mathcal{X}_{[0, U^0(\boldsymbol{x} - \boldsymbol{F}'(s)t)]}(s)$ solves the transport equation

$$\partial_t g + A^j(s) \partial_{x_j} g = 0, \qquad g(\boldsymbol{x}, s, 0) = \mathcal{X}_{[0, U^0(\boldsymbol{x})]}(s) \tag{8.2}$$

with $A^j = (F^j)'$ and $s \in \mathbb{R}$. Note that (8.2) is not a free transport equation. Instead, the flow velocity of the kinetic particles is given by $\boldsymbol{F}'(s)$. Compared to the original kinetic formulation (3.2) of the conservation law, the kinetic variable s seems to be somewhat artificial since its dimension does not match the dimension of $\boldsymbol{x}$. In view of the above derivation, however, the relation to the standard kinetic formulation based on the free transport equation is clarified. In particular, s is related to the state space integration occurring in the definition of the constraint function (8.1).

In [5, 6] it is shown that a kinetic scheme based on (8.2) (and hence on (8.1)) converges to the unique entropy solution of the Cauchy problem $\partial_t U + \partial_{x_j} F^j(U) = 0$. Also in [27] and [24] the relation between a transport equation of type (8.2) and the conservation law has been analyzed. In particular, it turns out that the kinetic approximation $U^{\circ}_{\mathring{\mu}}$ is the exact solution of the conservation law for small times, in accordance with Theorem 4.1.

9. Systems in one space dimension.

9.1. Strictly hyperbolic systems. Assuming strict hyperbolicity, the first step in the calculation of $\mathring{\boldsymbol{\mu}}$ is always to diagonalize $A = \nabla \boldsymbol{F}$ or, using the notation from [18], to write

$$A(\boldsymbol{U}) = \sum_{j=1}^{m} \lambda_j(\boldsymbol{U}) P_j(\boldsymbol{U}).$$

Here $P_j(\boldsymbol{U})$ is the projection on the jth eigenspace of $A(\boldsymbol{U})$. For the exponential of $-i\xi A$ we get

$$\exp(-i\xi A) = \sum_{j=1}^{m} e^{-i\xi\lambda_j} P_j$$

and hence

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}; v) = \int_{\Gamma_{\boldsymbol{U}}} \sum_{j=1}^{m} \delta(v - \lambda_j) P_j = \int_0^1 \sum_{j=1}^{m} \delta(v - \lambda_j(\boldsymbol{\gamma})) P_j(\boldsymbol{\gamma}) \dot{\boldsymbol{\gamma}}\, ds. \tag{9.1}$$

Obviously, the path integral simplifies a lot if $\dot{\boldsymbol{\gamma}}$ is piece-wise parallel to right eigenvectors $\boldsymbol{r}_i$ of A. In particular, if $\boldsymbol{r}_i \cdot \nabla \lambda_i$ does not change sign along the curve parallel to $\boldsymbol{r}_i$ (i.e., if the ith characteristic field is genuinely nonlinear), the line integral can be simplified further. It then leads to explicit expressions involving characteristic functions $\mathcal{X}$ as primitives of the δ distribution. For example, in the scalar case, genuine nonlinearity amounts to the condition $F'' > 0$ and we find

$$\mathring{\mu}(U; v) = \frac{1}{F''(s)} \mathcal{X}_{[F'(0), F'(U)]}(v), \qquad s = (F')^{-1}(v) \tag{9.2}$$

which is the constraint function obtained with a different approach in [2].

For the other extreme of linear degeneracy of the ith field (i.e., λ_i is constant along the $\boldsymbol{r}_i$-curve), the curve integral leads to a contribution proportional to $\delta(v - \lambda_i)$. We refer to [20] for further details. Some explicit examples are also given in sections 9.4 and 9.5.

9.2. The one-dimensional Euler system. In this example the state space $\mathcal{S}$ is three-dimensional. The vector of unknowns consists of mass density ρ, momentum density m, and energy density $\rho\epsilon$. Important derived quantities are velocity $u = m/\rho$, temperature $T = (\gamma - 1)(\epsilon - u^2/2)$, and pressure $p = \rho T$, where $\gamma > 1$ is a material constant. The state space $\mathcal{S}$ is a convex cone $\mathcal{S} = \{\rho(1, u, \epsilon)^t | \rho > 0, T > 0\}$. The nonlinear flux $\boldsymbol{F}$ is homogeneous of degree one so that its Jacobian A is homogeneous of degree zero

$$\boldsymbol{F} = \begin{pmatrix} \rho u \\ \rho(u^2 + T) \\ \rho(\epsilon + T)u \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma - 3)u^2 & (3 - \gamma)u & \gamma - 1 \\ \frac{1}{2}(\gamma - 2)u^3 - \frac{\gamma}{\gamma - 1} T u & (\frac{3}{2} - \gamma)u^2 + \frac{\gamma}{\gamma - 1} T & \gamma u \end{pmatrix}.$$

According to Theorem 4.1, the order of consistency of the kinetic scheme can only be higher than one if A^2 is exact. It turns out that the first row of A^2 is exact, the second row is exact only in the case $\gamma = 3$, but then the third row is not exact so that the kinetic scheme based on $\mathring{\boldsymbol{\mu}}$ is only first-order accurate and depends on the

selected curves $\Gamma_{\boldsymbol{U}}$ in state space. A choice which is motivated by the structure of $\mathcal{S}$ and $\boldsymbol{F}$ are straight lines

$$\Gamma_{\boldsymbol{U}} := \{s\boldsymbol{U} | s \in (0,1]\} \qquad \boldsymbol{U} \in \mathcal{S}.$$

On these curves, the Jacobian A is constant due to homogeneity of $\boldsymbol{F}$ so that

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}; v) = \int_{\Gamma_{\boldsymbol{U}}} \mathcal{F}_\xi^{-1} \exp(-i\xi A) = \mathcal{F}_\xi^{-1} \exp(-i\xi A(\boldsymbol{U}))\boldsymbol{U}. \tag{9.3}$$

To calculate $\mathcal{F}_\xi^{-1} \exp(-i\xi A)$ we diagonalize A which has eigenvalues $\lambda_1 = u$, $\lambda_2 = u - c$ and $\lambda_3 = u + c$ with the sound speed $c = \sqrt{\gamma T}$. In a basis of right eigenvectors, the matrix $\exp(-i\xi A)$ has the form $\operatorname{diag}(\exp(-i\xi\lambda_k))$ so that the inverse Fourier transform yields a linear superposition of $\delta(v - \lambda_k)$. Using the abbreviation

$$f(\boldsymbol{U}; v) = \rho\left(\frac{\gamma - 1}{\gamma}\delta(v - u) + \frac{1}{2\gamma}\delta(v - (u - c)) + \frac{1}{2\gamma}\delta(v - (u + c))\right)$$

the resulting constraint function can be written as

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}; v) = \begin{pmatrix} 1 \\ v \\ \frac{1}{2}v^2 \end{pmatrix} f(\boldsymbol{U}; v) + \left(\frac{1}{\gamma - 1} - \frac{1}{2}\right)\begin{pmatrix} 0 \\ 0 \\ |v - u|^2 \end{pmatrix} f(\boldsymbol{U}; v).$$

We remark that the same constraint function follows from the approach in [18]. We also note that, as in our introductory example (3.4), $\mathring{\boldsymbol{\mu}}$ is actually based on a nonnegative, scalar function f.

To analyze stability of the scheme we calculate the diffusion matrix $Q(\boldsymbol{U})$ given in Definition 7.1

$$Q(\boldsymbol{U}) = \begin{pmatrix} 0 & 0 & 0 \\ (\gamma - 3)Tu & (3 - \gamma)T & 0 \\ \frac{\gamma^2 - 9\gamma + 6}{2(\gamma - 1)}Tu^2 - \frac{\gamma}{\gamma - 1}T^2 & \frac{-2\gamma^2 + 5\gamma - 3}{\gamma - 1}Tu & \gamma T \end{pmatrix}.$$

The eigenvalues are $0, (3 - \gamma)T$ and γT so that the kinetic scheme is stable (in the sense of Definition 7.1) if $1 < \gamma \leq 3$.

9.3. A general class of 2×2 systems. We consider a class of hyperbolic systems where $\boldsymbol{U}$ ranges in a simply connected subset $\mathcal{S}$ of $\mathbb{R}^2$ and where $\boldsymbol{F} : \mathcal{S} \mapsto \mathbb{R}^2$ can be written as gradient of a smooth function $\phi : \mathcal{S} \mapsto \mathbb{R}$. Note that $\nabla \boldsymbol{F}$ is then the Hessian of ϕ and thus symmetric which implies hyperbolicity of the system.

For such systems it is possible to show that the maximal order of consistency of the kinetic scheme is *either one or infinity*, or in other words, if $\operatorname{con}(\mathring{\boldsymbol{\mu}}) > 1$, then $\operatorname{con}(\mathring{\boldsymbol{\mu}}) = \infty$. A system leading to infinite order is either decoupled (i.e., $\partial^2\phi/\partial U_1 \partial U_2 = 0$) or ϕ is a solution of the linear hyperbolic equation

$$\frac{\partial^2 \phi}{\partial U_1^2} + 2\gamma \frac{\partial^2 \phi}{\partial U_1 \partial U_2} - \frac{\partial^2 \phi}{\partial U_2^2} = 0 \tag{9.4}$$

for some $\gamma \in \mathbb{R}$. This relation for ϕ follows from the condition that A^2 is exact. For the Jacobian $A = \nabla \boldsymbol{F}$ it implies the structure

$$A(\boldsymbol{U}) = \begin{pmatrix} a(\boldsymbol{U}) & b(\boldsymbol{U}) \\ b(\boldsymbol{U}) & a(\boldsymbol{U}) + 2\gamma b(\boldsymbol{U}) \end{pmatrix}$$

giving rise to

$$\exp(-i\xi A) = \frac{e^{-i\xi a}}{1+\lambda_+^2}\begin{pmatrix} e^{-i\xi\lambda_+ b}+\lambda_+^2 e^{-i\xi\lambda_- b} & \lambda_+(e^{-i\xi\lambda_+ b}-e^{-i\xi\lambda_- b}) \\ \lambda_+(e^{-i\xi\lambda_+ b}-e^{-i\xi\lambda_- b}) & \lambda_+^2 e^{-i\xi\lambda_+ b}+e^{-i\xi\lambda_- b}\end{pmatrix}, \tag{9.5}$$

where $\lambda_\pm = \gamma \pm \sqrt{1+\gamma^2}$. The exactness of (9.5) for arbitrary smooth functions a, b can then be checked directly by showing

$$\frac{\partial}{\partial U_k}\exp(-i\xi A(\boldsymbol{U}))_{nl} = \frac{\partial}{\partial U_l}\exp(-i\xi A(\boldsymbol{U}))_{nk}, \quad k,l,n \in \{1,2\}.$$

Altogether, a large class of systems giving rise to infinite order of consistency can be constructed based on solutions ϕ of (9.4).

As we will see below, also the examples on the isentropic Euler system in one and two space dimensions as well as the p-system show that the order of consistency is either one or infinity. Whether this property holds for general systems remains to be investigated.

9.4. Isentropic Euler equations in one dimension. Under smooth conditions, the isentropic Euler equations can be derived from the system in section 9.2 because the energy equation can be transformed into an equation for entropy which decouples from the system if the entropy is initially constant. The pressure p which appears as source term in the momentum equation depends only on mass density and the constant value of entropy. As vector of unknowns, we now have $\boldsymbol{U} = (\rho, m)^t$ with $\mathcal{S} = \{(\rho,m)^t | \rho > 0, m \in \mathbb{R}\}$ and

$$\boldsymbol{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho)\end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -u^2+p' & 2u\end{pmatrix}, \qquad u = \frac{m}{\rho}.$$

The system is hyperbolic if $p' \geq 0$ and one can show that infinite order of consistency is obtained if $p(\rho)/\rho = p''(\rho)/2$, which singles out the pressure laws $p(\rho) = C\rho^3 + D$. In all other cases, the kinetic scheme will be first-order consistent.

Since we consider the case of constant pressure $p(\rho) = D$ in detail for the two-dimensional isentropic Euler system in section 10.2, we just mention here that it leads to $\mathring{\boldsymbol{\mu}} = \rho\binom{1}{v}\delta(v-u)$. This constraint function has been used by other authors to show the relation between the constant pressure system and kinetic theory (see [9] and the references therein).

In the following, we restrict ourselves to the case $c = \sqrt{p'} > 0$ and $p'' > 0$. The exponential $\hat{E}$ is given by

$$\exp(-i\xi A) = \begin{pmatrix} 1 \\ i\partial_\xi\end{pmatrix}\frac{e^{-i\xi u}}{c}\begin{pmatrix} c\cos(c\xi)+iu\sin(c\xi) & -i\sin(c\xi)\end{pmatrix}. \tag{9.6}$$

To obtain $\mathring{\boldsymbol{\mu}}$, we integrate (9.6) over suitable families of curves and take the inverse Fourier transform with respect to ξ. More precisely, we will investigate different families of curves. Of course,

$$\Gamma_{\boldsymbol{U}} = \{s\boldsymbol{U} | s \in (0,1]\}, \qquad \boldsymbol{U} \in \mathcal{S} \tag{9.7}$$

is again a reasonable choice, but we will also consider integral curves of the right eigenvectors $\boldsymbol{r}_1 = \frac{\rho}{2c}\binom{1}{u-c}$ and $\boldsymbol{r}_2 = \frac{\rho}{2c}\binom{1}{u+c}$ of A, or equivalently coordinate lines of the system of Riemann invariants [31]

$$\begin{pmatrix} W_1 \\ W_2\end{pmatrix} = \begin{pmatrix} \tilde{c}(\rho) - u \\ \tilde{c}(\rho) + u\end{pmatrix}, \qquad \tilde{c}(\rho) = \int_0^\rho \frac{c(\sigma)}{\sigma}\, d\sigma.$$

For the calculation of $\mathring{\boldsymbol{\mu}}$ along $\boldsymbol{r}_1$-curves we first need a parameterization $\boldsymbol{\gamma}(\boldsymbol{U}, s)$. By definition, $\boldsymbol{U} \in \mathcal{S}$ is located on the $\boldsymbol{r}_1$-curve corresponding to $W_2 = W_2(\boldsymbol{U})$. Therefore, a simple parameterization is given by

$$\gamma(\boldsymbol{U}, s) := \begin{pmatrix} s\rho \\ s\rho(W_2(\boldsymbol{U}) - \tilde{c}(s\rho)) \end{pmatrix} \qquad s \in (0, 1].$$

Completing the calculation, we obtain

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}; v) = \begin{pmatrix} 1 \\ v \end{pmatrix} \left[(\tilde{c} + c)^{-1}\right]' (u - v + \tilde{c}(\rho)) \mathcal{X}_{[-c(\rho), \tilde{c}(\rho)]}(v - u).$$

Note that the v-support of $\mathring{\boldsymbol{\mu}}$ is in general not symmetric with respect to u. Only in the case $c = \tilde{c}$, which is equivalent to $p = C\rho^3$, we get symmetry.

Using the $\boldsymbol{r}_2$-curves we obtain in a completely analogous manner

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}; v) = \begin{pmatrix} 1 \\ v \end{pmatrix} \left[(\tilde{c} + c)^{-1}\right]' (v - u + \tilde{c}(\rho)) \mathcal{X}_{[-\tilde{c}(\rho), c(\rho)]}(v - u).$$

The support is again asymmetric, but the role of $\tilde{c}$ and c has exchanged.

A symmetric distribution is obtained with the curves (9.7) which are in some sense a compromise between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$-curves. Indeed, the tangent vector $\rho \binom{1}{u}$ is, up to the factor $1/2c$, just the average $\frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$. We obtain

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}; v) = \begin{pmatrix} 1 \\ v \end{pmatrix} \left[c^{-1}\right]' (|v - u|) \mathcal{X}_{[0, c(\rho)]}(|v - u|).$$

This constraint function has been derived in [22] with a different approach.

We remark that in the particular case $\gamma = 3$, all the constraint functions coincide because the chosen path of integration does not influence the value of the integral. Also, $c(\rho) = \tilde{c}(\rho) = \sqrt{3C}\rho$ so that $[c^{-1}]'$ is a constant and $\mathring{\boldsymbol{\mu}}$ is determined by the scalar constraint function

$$f(\boldsymbol{U}; v) = \frac{1}{\sqrt{3C}} \mathcal{X}_{[0, c(\rho)]}(|v - u|).$$

Recently, a hierarchy of hyperbolic systems has been derived which includes the isentropic Euler system with $\gamma = 3$ as a special case [8]. Other systems in the hierarchy involve a higher dimensional state space $\mathcal{S}$. By construction, smooth solutions of these systems can be written in the form of the kinetic scheme with a particular constraint function. Since exactness of the solution implies infinite order of consistency of the kinetic scheme, we conclude with Theorem 4.1 that the constraint functions used in [8] essentially coincide with those introduced here.

To complete the example, we take a look at stability. For each of the chosen families of curves one can show that the diffusion matrix has the form

$$Q(\boldsymbol{U}) = \left(3\frac{p}{\rho} - p'\right) \begin{pmatrix} 0 & 0 \\ * & 1 \end{pmatrix},$$

where the entry $*$ is equal to $-u$ for the straight lines (9.7), equal to $c - u$ for the $\boldsymbol{r}_1$-curves, and $-(u + c)$ for the $\boldsymbol{r}_2$-curves. Consequently, the kinetic schemes are stable if and only if $3p/\rho - p' \geq 0$. In the case $p = C\rho^\gamma$ this is equivalent to $\gamma \leq 3$.

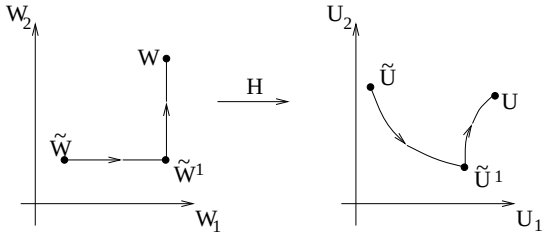

FIG. 9.1. *Curve along characteristic fields.*

9.5. The *p*-system. In this example, we consider the nonlinear wave equation

$$\partial_t^2\varphi + \partial_x^2 p(\varphi) = 0, \qquad p' < 0, p'' > 0. \tag{9.8}$$

Motivated by the linear case $p(\varphi) = -c^2\varphi$ we introduce the wave speed $c(\varphi) = \sqrt{-p'(\varphi)}$. As an important example, we mention the isentropic Euler equation in Lagrangian coordinates. Here φ is interpreted as specific volume $1/\rho$ and p is the pressure which typically decreases with decreasing ρ. To treat equation (9.8) in our context we first transform it into a system of first-order equations, the so-called p-system [31]. Setting $U_1 = \varphi$ which ranges in some interval $I \subset \mathbb{R}$ and $U_2 = \partial_t\varphi \in \mathbb{R}$, we obtain a system with flux vector

$$\boldsymbol{F} = \begin{pmatrix} -U_2 \\ p(U_1) \end{pmatrix}, \qquad \boldsymbol{U} \in \mathcal{S} = I \times \mathbb{R}.$$

For the Jacobian $A = \nabla\boldsymbol{F}$, we find the exponential matrix

$$\exp(-i\xi A) = \begin{pmatrix} 1 \\ -i\partial_\xi \end{pmatrix} \begin{pmatrix} \cos(\xi c) & \frac{1}{c} i \sin(\xi c) \end{pmatrix}$$

which is exact only in the linear case where c is independent of U_1. In all other cases, the kinetic scheme is first-order consistent and properties of $\mathring{\boldsymbol{\mu}}$ are influenced by the chosen curves $\Gamma_{\boldsymbol{U}}$.

In general, the state space $\mathcal{S}$ will not possess a distinguished point like the origin in the previous examples. We thus pick any $\tilde{U}_1 \in I$ and use $\tilde{\boldsymbol{U}} = \left(\begin{smallmatrix} \tilde{U}_1 \\ 0 \end{smallmatrix}\right)$ as a starting point for the family of curves which we again take as integral lines of the right eigenvectors or, equivalently, as piecewise coordinate lines of a system of Riemann invariants (see Figure 9.1). As mapping $\boldsymbol{H}$ from the Riemann invariants $\boldsymbol{W}$ to the conservative variables $\boldsymbol{U}$ we choose

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} U_2 + \bar{c}(U_1) \\ U_2 - \bar{c}(U_1) \end{pmatrix}, \qquad \bar{c} = \int_{\tilde{U}_1}^{U_1} c(\sigma)\, d\sigma,$$

such that the image of the reference state $\tilde{\boldsymbol{U}}$ is just $\tilde{\boldsymbol{W}} = \boldsymbol{0}$. The point where the curve switches from the first to the second field is denoted $\tilde{\boldsymbol{W}}^1$, respectively, $\tilde{\boldsymbol{U}}^1 = \boldsymbol{H}(\tilde{\boldsymbol{W}}^1)$. Calculating the curve integral over $E = \mathcal{F}_\xi^{-1} \exp(-i\xi A)$, we eventually find

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}; v) = \left[c^{-1}\right]'(|v|) \begin{pmatrix} 1 \\ -v \end{pmatrix} \left(\mathcal{X}_{[-c(\tilde{U}_1^1), c(\tilde{U}^1)]}(v) + \mathcal{X}_{[c(\tilde{U}_1^1), c(U_1)]}(v) \right).$$

Note that the arbitrary point $\tilde{\boldsymbol{U}}$ decisively determines the support of $\mathring{\boldsymbol{\mu}}$. In fact, this arbitrariness can be the reason for instability. To see this, we calculate the diffusion

matrix

$$Q(\boldsymbol{U}) = \Big(c(\tilde{U}_1^1) - c(U_1)\Big) \begin{pmatrix} 0 & 0 \\ c(U_1) & 1 \end{pmatrix}$$

which has a negative eigenvalue if $U_1 < \tilde{U}_1^1$ (note that c is decreasing due to $p'' > 0$). Consequently, the state space splits into a stable and an unstable region which are separated by the $\boldsymbol{r}_2$-curve through $\tilde{\boldsymbol{U}}$.

10. A two-dimensional system. As an example, we consider the isentropic Euler system in two space dimensions. To get hyperbolicity we restrict ourselves to pressure functions which satisfy $p' \geq 0$. The vector of unknowns $\boldsymbol{U} = (\rho, m_1, m_2)^t$ consists of density ρ and momentum $\boldsymbol{m}$ with $\mathcal{S} = \{(\rho, m_1, m_2)^t | \rho > 0, \boldsymbol{m} \in \mathbb{R}^2\}$ being a convex cone. Using again $\boldsymbol{u} = \boldsymbol{m}/\rho$, the fluxes are of the form

$$\boldsymbol{F}^1 = \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p(\rho) \\ \rho u_1 u_2 \end{pmatrix}, \qquad \boldsymbol{F}^2 = \begin{pmatrix} \rho u_2 \\ \rho u_1 u_2 \\ \rho u_2^2 + p(\rho) \end{pmatrix}$$

with Jacobians

$$A^1 = \begin{pmatrix} 0 & 1 & 0 \\ -u_1^2 + p'(\rho) & 2u_1 & 0 \\ -u_1 u_2 & u_2 & u_1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 1 \\ -u_1 u_2 & u_2 & u_1 \\ -u_2^2 + p'(\rho) & 0 & 2u_2 \end{pmatrix}.$$

To see whether the kinetic scheme based on $\mathring{\boldsymbol{\mu}}$ can be second-order accurate we check the exactness of the products A^1A^1, A^2A^2, A^1A^2, and A^2A^1. In all cases, the first rows are exact but in the second and third rows we find nontrivial conditions

$$\frac{p'}{\rho} = \frac{1}{2}p'', \qquad \frac{p'}{\rho} = 0, \qquad p'' = 0.$$

These conditions are simultaneously only satisfied in the case of constant pressure and, as we shall see in section 10.2, the kinetic scheme based on $\mathring{\boldsymbol{\mu}}$ then leads to infinite order of accuracy. For all other pressure laws, the kinetic scheme is always first-order accurate.

10.1. Nonconstant pressure laws. Choosing again $\Gamma_{\boldsymbol{U}} = \{s\boldsymbol{U} | s \in (0, 1]\}$, we find after transformation to a basis of eigenvectors of $\xi_j A^j$ and integration

$$\int_{\Gamma_{\boldsymbol{U}}} \exp(-i\xi_j A^j) = \rho \begin{pmatrix} 1 \\ i\partial_{\xi_1} \\ i\partial_{\xi_2} \end{pmatrix} e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}} \int_0^1 \cos(c(s\rho)|\boldsymbol{\xi}|)\, ds. \tag{10.1}$$

Under the additional assumptions

$$c(0) = 0, \qquad c' > 0, \qquad \lim_{\sigma \to 0} \frac{\sigma}{c'(c^{-1}(\sigma))} = 0,$$

which are satisfied for the practically relevant pressure laws $p(\rho) = C\rho^\gamma$ with $1 < \gamma \leq 3$ we can calculate the Fourier transform of (10.1) and get [20]

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U}; \boldsymbol{v}) = \begin{pmatrix} 1 \\ v_1 \\ v_2 \end{pmatrix} a(\rho, |\boldsymbol{v} - \boldsymbol{u}|) \mathcal{X}_{[0, c(\rho))}(|\boldsymbol{v} - \boldsymbol{u}|).$$

We remark that the constraint function has again a structure similar to our initial example (3.4). In fact it consists of a scalar density

$$f(\boldsymbol{U};\boldsymbol{v}) = a(\rho,|\boldsymbol{v}-\boldsymbol{u}|)\mathcal{X}_{[0,c(\rho))}(|\boldsymbol{v}-\boldsymbol{u}|) \tag{10.2}$$

which is multiplied by the vector $\binom{1}{\boldsymbol{v}}$. For pressure laws $p(\rho) = C\rho^\gamma$, the function a is of the form

$$a_\gamma(\rho,|\boldsymbol{v}|) = \frac{\rho}{\pi(\gamma-1)c^2(\rho)}\left(\frac{1}{\sqrt{1-\nu^2}} - \frac{3-\gamma}{\gamma-1}\int_\nu^1 \frac{\sigma^{\frac{4-2\gamma}{\gamma-1}}}{\sqrt{\sigma^2-\nu^2}}\,d\sigma\right)$$

with $\nu = |\boldsymbol{v}|/c(\rho)$. Physically, f in (10.2) is interpreted as particle distribution function so that a natural property would be $f \geq 0$. It turns out, however, that a_γ and thus also f is not always nonnegative. For example, for $\gamma = 7/5$ we have

$$a_{7/5}(\rho,|\boldsymbol{v}|) = \frac{5\rho}{2\pi c^2(\rho)}\frac{8\nu^4-4\nu^2-1}{\sqrt{1-\nu^2}}$$

which changes sign at $\nu = \frac{1}{2}\sqrt{1+\sqrt{3}}$. In fact, one can show in this case that any distribution function of the form (10.2) with $|\boldsymbol{v}|$ support in $[0,c(\rho))$ has to be negative for some $\boldsymbol{v}$ [20]. To avoid this behavior, the support has to be larger as for example in [22].

10.2. The case of constant pressure. In this particular case, the linear combination $\xi_j A^j$ can only be transformed into a Jordan matrix and $\hat{E}$ has the form

$$\exp(-i\xi_j A^j) = e^{-i(u,\xi)}\begin{pmatrix} 1+i(\xi,u) & -i\xi_1 & -i\xi_2 \\ i(\xi,u)u_1 & 1-i\xi_1u_1 & -i\xi_2u_1 \\ i(\xi,u)u_2 & -i\xi_1u_2 & 1-i\xi_2u_2 \end{pmatrix}.$$

An easy but lengthy calculation shows that $\exp(-i\xi_j A^j)$ is exact for any $\boldsymbol{\xi} \in \mathbb{R}^2$. Consequently, $\mathring{\boldsymbol{\mu}}$ is independent of the chosen path and the resulting kinetic scheme is linearly stable. To carry out the integration, we choose $\Gamma_{\boldsymbol{U}} = \{s\boldsymbol{U}|s \in (0,1]\}$ since $\exp(-i\xi_j A^j)$ is constant along these paths. We find

$$\int_{\Gamma_{\boldsymbol{U}}} \exp(-i\xi_j A^j) = \exp(-i\xi_j A^j(\boldsymbol{U}))\boldsymbol{U} = \rho\begin{pmatrix}1\\ \boldsymbol{u}\end{pmatrix} e^{-i\boldsymbol{u}\cdot\boldsymbol{\xi}}$$

so that the Fourier transform is given by

$$\mathring{\boldsymbol{\mu}}(\boldsymbol{U};\boldsymbol{v}) = \rho\begin{pmatrix}1\\ v_1\\ v_2\end{pmatrix}\delta(\boldsymbol{v}-\boldsymbol{u}). \tag{10.3}$$

In [20] it is shown that for smooth initial values and small times the kinetic scheme based on (10.3) yields the exact solution of the problem which is in accordance with the infinite order of consistency.

11. Relation to Brenier's method. The construction of the constraint function $\mathring{\boldsymbol{\mu}}$ is based on a curve integral of the matrix $E(\boldsymbol{U}) = \mathcal{F}_{\boldsymbol{\xi}}^{-1}\big(\exp(-i\xi_j A^j(\boldsymbol{U}))\big)$. However, if $E(\boldsymbol{U})$ is exact, the constraint $\mathring{\boldsymbol{\mu}}$ is independent of the chosen family of curves $\boldsymbol{\gamma}$ which motivates a formulation without explicit reference to $\boldsymbol{\gamma}$. Such a reformulation is indeed possible, following the approach for the strictly hyperbolic,

one-dimensional case in [7]. To avoid the restriction $d = 1$, we slightly modify the argument. Also, our considerations here are purely formal for brevity.

Let us assume that $\boldsymbol{U}^0$ is a smooth initial value which is constant at large $\boldsymbol{x}$, i.e., $\boldsymbol{U}^0(\boldsymbol{x}) = \boldsymbol{U}_\infty$ for $|\boldsymbol{x}| > R$. If the matrix $E(\boldsymbol{U})$ is exact, we have $\nabla \mathring{\boldsymbol{\mu}} = E$ so that

$$\nabla_{\boldsymbol{x}} \boldsymbol{U}^{\circ}_{\mathring{\boldsymbol{\mu}}}(t, \boldsymbol{x}) = \left\langle E(\boldsymbol{U}^0(\boldsymbol{x} - t\boldsymbol{v}); \boldsymbol{v}) \nabla_{\boldsymbol{x}} \boldsymbol{U}^0(\boldsymbol{x} - t\boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}. \tag{11.1}$$

Using the fact that $\delta(\boldsymbol{y}) = \operatorname{div}\left(\frac{\boldsymbol{y}}{\omega_d |\boldsymbol{y}|^d}\right)$, where ω_d is the area of the unit sphere in d space dimensions, we can write

$$\boldsymbol{U}^{\circ}_{\mathring{\boldsymbol{\mu}}}(t, \boldsymbol{x}) = \boldsymbol{U}_\infty + \left\langle \delta(\boldsymbol{y}), \boldsymbol{U}^{\circ}_{\mathring{\boldsymbol{\mu}}}(t, \boldsymbol{x} - \boldsymbol{y}) \right\rangle_{\boldsymbol{y}} = \boldsymbol{U}_\infty + \left\langle \frac{\boldsymbol{y}}{\omega_d |\boldsymbol{y}|^d}, \nabla_{\boldsymbol{x}} \boldsymbol{U}^{\circ}_{\mathring{\boldsymbol{\mu}}}(t, \boldsymbol{x} - \boldsymbol{y}) \right\rangle_{\boldsymbol{y}}.$$

Now (11.1) implies

$$\boldsymbol{U}^{\circ}_{\mathring{\boldsymbol{\mu}}}(t, \boldsymbol{x}) = \boldsymbol{U}_\infty + \left\langle \frac{\boldsymbol{y}}{\omega_d |\boldsymbol{y}|^d}, \left\langle E(\boldsymbol{U}^0(\boldsymbol{x} - \boldsymbol{y} - t\boldsymbol{v}); \boldsymbol{v}) \nabla_{\boldsymbol{x}} \boldsymbol{U}^0(\boldsymbol{x} - \boldsymbol{y} - t\boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}} \right\rangle_{\boldsymbol{y}}$$

or with the change of variables $\boldsymbol{x} - \boldsymbol{y} - t\boldsymbol{v} = \boldsymbol{\xi}$

$$\tilde{\boldsymbol{U}}^{\circ}_{\mathring{\boldsymbol{\mu}}}(t, \boldsymbol{x}) = \boldsymbol{U}_\infty + \left\langle \frac{\boldsymbol{x} - \boldsymbol{\xi} - t\boldsymbol{v}}{\omega_d |\boldsymbol{x} - \boldsymbol{\xi} - t\boldsymbol{v}|^d} E(\boldsymbol{U}^0(\boldsymbol{\xi}); \boldsymbol{v}) \nabla_{\boldsymbol{x}} \boldsymbol{U}^0(\boldsymbol{\xi}), 1 \right\rangle_{(\boldsymbol{v}, \boldsymbol{\xi})}. \tag{11.2}$$

(We introduce the new notation $\tilde{\boldsymbol{U}}^{\circ}_{\mathring{\boldsymbol{\mu}}}$ since (11.2) coincides with the kinetic scheme only in the case $\operatorname{con}(\mathring{\boldsymbol{\mu}}) = \infty$—otherwise, it is a different approximation). To see that (11.2) is an extension of Brenier's method, we apply it to the strictly hyperbolic case with $d = 1$, where, according to section 9.1, $E = \sum_k \delta(v - \lambda_k) P_k$. Denoting the projections of the space derivative of the initial value as $\tilde{\boldsymbol{r}}_k$, relation (11.2) reduces to

$$\tilde{\boldsymbol{U}}^{\circ}_{\mathring{\boldsymbol{\mu}}}(t, x) = \boldsymbol{U}_\infty + \int_{\mathbb{R}} \sum_{k=1}^{m} \frac{1}{2} \operatorname{sign}(x - \xi - \lambda_k(\boldsymbol{U}^0(\xi)) t) \tilde{\boldsymbol{r}}_k(\xi)\, d\xi.$$

The same expression is obtained if the relation $\delta(y) = H'(y)$ used in the derivation in [7] is modified to $\delta(y) = \operatorname{sign}'(y)/2$. We remark, however, that (11.2) is also applicable to multidimensional problems and nonstrictly hyperbolic systems. It turns out that (11.2) is, in general, a first-order consistent approximation to the hyperbolic system (the consistency can easily be checked with relations (5.6) and (5.7)). Higher order consistency is only obtained if $\operatorname{con}(\mathring{\boldsymbol{\mu}}) \geq 2$ or $d = 1$ (for the latter case see [23]). Whether the extension (11.2) of Brenier's method leads to applicable numerical schemes similar to the case $d = 1$ (see [7, 11]) remains to be investigated.

12. Conclusion. We have presented a general construction principle for constraint functions used in kinetic schemes which opens the approach to general hyperbolic systems. The principle extends and generalizes several concepts proposed by other authors. Moreover, a specific criterion is presented which singles out a certain class of equations for which the kinetic approach is particularly well suited. This class includes all linear hyperbolic systems, nonlinear scalar equations, as well as some nonlinear systems.

Appendix. Structure of the constraint function. The aim of the appendix is to describe the mathematical structure of the constraint function $\mathring{\boldsymbol{\mu}}$. At the same

time, we are going to extend the concept of kinetic formulations to entropy conservation laws related to the system (3.1). Here a convex scalar function $\eta : \mathcal{S} \mapsto \mathbb{R}$ is an entropy function with entropy fluxes $\varphi^j : \mathcal{S} \mapsto \mathbb{R}$ provided

$$\nabla^t \eta \nabla \boldsymbol{F}^j = \nabla^t \varphi^j, \qquad j = 1, \dots, d, \tag{A.1}$$

where $\nabla^t \eta = (\nabla \eta)^t$. Of course, differentiability of η and φ^j is required. We will also assume that $\eta(\mathbf{0}) = 0$ as well as $\varphi^j(\mathbf{0}) = 0$ which can be achieved by subtracting the value at zero. If $\boldsymbol{U}$ is a smooth solution of (3.1), relation (A.1) implies that $\eta(\boldsymbol{U})$ satisfies an additional conservation law

$$\partial_t \eta(\boldsymbol{U}) + \partial_{x_j} \varphi^j(\boldsymbol{U}) = 0. \tag{A.2}$$

A kinetic formulation of the conservation law (A.2) is obtained with an *entropy constraint function* $\mu_\eta(\boldsymbol{U}; \boldsymbol{v})$ which satisfies the consistency condition

$$\langle \mu_\eta(\boldsymbol{U}), 1 \rangle_{\boldsymbol{v}} = \eta(\boldsymbol{U}), \qquad \langle \mu_\eta(\boldsymbol{U}), v_j \rangle_{\boldsymbol{v}} = \varphi^j(\boldsymbol{U}). \tag{A.3}$$

Then (A.2) follows from the constrained evolution

$$\partial_t \Phi + v_j \partial_{x_j} \Phi = q, \qquad \Phi(t, \boldsymbol{x}, \boldsymbol{v}) = \mu_\eta(\boldsymbol{U}(t, \boldsymbol{x}); \boldsymbol{v}), \qquad \langle q, 1 \rangle_{\boldsymbol{v}} = 0.$$

Note that our original considerations are included in this approach by choosing special linear entropy-entropy flux pairs

$$\eta(\boldsymbol{U}) = U_i, \qquad \varphi^j(\boldsymbol{U}) = F_i^j(\boldsymbol{U}), \qquad i \in \{1, \dots, d\}. \tag{A.4}$$

With this choice, which clearly satisfies relation (A.1), the conservation law (A.2) is just the ith member of the system of conservation laws (3.1). This observation enables us to investigate the constraint functions for the system (3.1) and the entropy constraint functions for (A.2) simultaneously.

As an entropy constraint function, we propose

$$\mathring{\mu}_\eta(\boldsymbol{U}) = \int_{\Gamma_{\boldsymbol{U}}} \nabla^t \eta E, \qquad E(\boldsymbol{U}) = \mathcal{F}_{\boldsymbol{\xi}}^{-1} \exp\left(-i \xi_j A^j(\boldsymbol{U})\right) \tag{A.5}$$

which reduces to (6.3) for the entropies (A.4) and satisfies (A.3). Indeed, using the fact that $\langle E, 1 \rangle = \mathrm{I}$ and $\langle E, v_k \rangle = A^k(\boldsymbol{U})$ for every $\boldsymbol{U} \in \mathcal{S}$, we get

$$\langle \mathring{\mu}_\eta(\boldsymbol{U}), 1 \rangle_{\boldsymbol{v}} = \int_{\Gamma_{\boldsymbol{U}}} \nabla^t \eta \, \langle E, 1 \rangle = \int_{\Gamma_{\boldsymbol{U}}} \nabla^t \eta = \eta(\boldsymbol{U})$$

and

$$\langle \mathring{\mu}_\eta(\boldsymbol{U}), v_k \rangle_{\boldsymbol{v}} = \int_{\Gamma_{\boldsymbol{U}}} \nabla^t \eta \, \langle E, v_k \rangle = \int_{\Gamma_{\boldsymbol{U}}} \nabla^t \eta A^k = \int_{\Gamma_{\boldsymbol{U}}} \nabla \varphi^k = \varphi^k(\boldsymbol{U}).$$

To describe the mathematical structure of (A.5), we begin with some remarks on the function

$$\hat{E}(\boldsymbol{U}; \boldsymbol{\xi}) = \exp\left(-i \xi_j A^j(\boldsymbol{U})\right), \qquad A^j = \nabla \boldsymbol{F}^j$$

(for details and proofs we refer to [20]). At the core of the investigations is a result from the theory of linear hyperbolic systems which is based on the following lemma [32].

LEMMA A.1. *Let M be any $m \times m$ matrix. There is a constant C_m depending only on m such that*

$$\|\exp(iM)\| \leq C_m\,(1 + \|M\|)^m \exp(I(M)),$$

where $I(M)$ is the largest absolute value of the imaginary parts of the eigenvalues of M.

If we take in particular $M = \xi_j A^j(\boldsymbol{U})$, hyperbolicity implies that all eigenvalues of M are real so that $I(M) = 0$. This leads to the estimate

$$\|\hat{E}(\boldsymbol{U};\boldsymbol{\xi})\| \leq C_{\boldsymbol{U}}(1 + |\boldsymbol{\xi}|)^m,$$

where $C_{\boldsymbol{U}}$ depends on $\max_{j=1}^m \|A^j(\boldsymbol{U})\|$. In particular, each component of the matrix $\hat{E}(\boldsymbol{U};\boldsymbol{\xi})$ grows at most polynomially in $|\boldsymbol{\xi}|$. Thus $\hat{E}(\boldsymbol{U})$ can be interpreted as a matrix of regular, tempered distributions on $\mathbb{R}^d$, i.e., $\hat{E}(\boldsymbol{U}) \in \mathcal{S}'(\mathbb{R}^d)^{m\times m}$. Moreover, $\boldsymbol{U} \mapsto \hat{E}(\boldsymbol{U})$ is continuous as a mapping from $\mathcal{S}$ to $[\mathcal{S}']^{m\times m}$. Since the $\boldsymbol{\xi}$ dependence of $\hat{E}$ is analytic, the Paley–Wiener theorem implies that for each $\boldsymbol{U} \in \mathcal{S}$ the inverse Fourier transform $E(\boldsymbol{U}) = \mathcal{F}^{-1}\hat{E}(\boldsymbol{U})$ is a matrix of compactly supported distributions. The mapping $\boldsymbol{U} \mapsto E(\boldsymbol{U})$ from $\mathcal{S}$ to $[\mathcal{E}']^{m\times m}$ is also continuous.

Our basic assumptions on the curves $\Gamma_{\boldsymbol{U}} \subset \mathcal{S}$ which appear in the definition of the constraint functions (A.5) are listed in the following definition. The requirements are satisfied, for example, by curves which consist of piecewise constant segments along the coordinate axes or smooth transformations thereof.

DEFINITION A.2 (admissible family of curves). *Let $\mathcal{S} \subset \mathbb{R}^m$ be open with $\mathbf{0} \in \bar{\mathcal{S}}$. A function $\boldsymbol{\gamma} : \mathcal{S}\times[0,1] \mapsto \bar{\mathcal{S}}$ is called an admissible family of curves in $\mathcal{S}$ if $\boldsymbol{\gamma}(\boldsymbol{U},0) = \mathbf{0}$, and $\boldsymbol{\gamma}(\boldsymbol{U},1) = \boldsymbol{U}$, if $\boldsymbol{\gamma} : \mathcal{S}\times[0,1] \mapsto \bar{\mathcal{S}}$ is continuous, and $\boldsymbol{\gamma}(\boldsymbol{U},\cdot) : [0,1] \mapsto \bar{\mathcal{S}}$ is piecewise C^1, if $\dot{\boldsymbol{\gamma}} \in \mathbb{L}^\infty_{loc}(\mathcal{S}\times[0,1],\mathbb{R}^m)$, and $\nabla\dot{\boldsymbol{\gamma}} \in \mathbb{L}^\infty_{loc}(\mathcal{S}\times[0,1],\mathbb{R}^m)$, where ∇ refers to the $\boldsymbol{U}$-derivative and the dot to the s-derivative. The image of $\boldsymbol{\gamma}(\boldsymbol{U},\cdot)$ is denoted $\Gamma_{\boldsymbol{U}}$.*

Since our main interest is to integrate $\nabla^t\eta E$ along the curves $\Gamma_{\boldsymbol{U}}$, we introduce suitable families of curves for this purpose.

DEFINITION A.3 ($\boldsymbol{F}$-admissible family). *Let $\boldsymbol{\gamma}$ be an admissible family in $\mathcal{S}$. A measurable function $\boldsymbol{f} : \mathcal{S} \mapsto \mathbb{R}^{m'}$ is called locally bounded on $\boldsymbol{\gamma}$ if for any compact set $K \subset \mathcal{S}$*

$$\|\boldsymbol{f}(\boldsymbol{\gamma})\|_{\mathbb{L}^\infty(K\times[0,1])} < \infty.$$

The family $\boldsymbol{\gamma}$ is called $\boldsymbol{F}$-admissible if the derivatives $A^j = \nabla\boldsymbol{F}^j$ are locally bounded on $\boldsymbol{\gamma}$.

We remark that Definition A.3 is only necessary if we want to include cases where the curves $\Gamma_{\boldsymbol{U}}$ touch $\partial\mathcal{S}$. Otherwise, the $\boldsymbol{F}$-admissibility is an immediate consequence of continuity of the matrices A^j.

In the following, we always assume that η is an entropy which is locally bounded on the $\boldsymbol{F}$-admissible family $\boldsymbol{\gamma}$. Then the linear mapping

$$\phi \mapsto \int_0^1 \langle \nabla^t\eta(\boldsymbol{\gamma}(\boldsymbol{U},s))E(\boldsymbol{\gamma}(\boldsymbol{U},s);\boldsymbol{v})\dot{\boldsymbol{\gamma}}(\boldsymbol{U},s), \phi(\boldsymbol{v})\rangle_{\boldsymbol{v}}\, ds, \qquad \phi \in \mathcal{E}(\mathbb{R}^d)$$

defines a compactly supported distribution $\mathring{\mu}_\eta(\boldsymbol{U}) \in \mathcal{E}'(\mathbb{R}^d)$ which depends continuously on $\boldsymbol{U}$. This result, which is proved in [20], justifies the notation (A.5). A careful analysis of the continuity properties of the mapping $\boldsymbol{U} \mapsto \mathring{\mu}_\eta(\boldsymbol{U}) \in \mathcal{E}'(\mathbb{R}^d)$ motivates

the following definition of a larger set of constraint functions which contains $\mathring{\mu}_\eta$ as a special case.

DEFINITION A.4. *By $\mathcal{K}$ we denote the set of all continuous $\mathcal{E}'(\mathbb{R}^d)$ valued functions μ on $\mathcal{S}$ which satisfy for any compact $K \subset \mathcal{S}$ and any $\phi \in \mathcal{E}(\mathbb{R}^d)$*

$$|\langle \mu(\boldsymbol{U};\boldsymbol{v}), \phi(\boldsymbol{v})\rangle_{\boldsymbol{v}}| \leq C_K \sum_{|\boldsymbol{\alpha}|\leq N} \sup_{|\boldsymbol{v}|\leq r_K} |D^{\boldsymbol{\alpha}}\phi(\boldsymbol{v})| \qquad \text{for all } \boldsymbol{U} \in K,$$

where $N \in \mathbb{N}$ depends only on μ and $C_K, r_K > 0$ are constants depending on μ and K.

For $\mu \in \mathcal{K}$, the operations that typically appear in the framework of kinetic schemes are justified. Under certain conditions on $\boldsymbol{U}^0$, for example, the expression

$$\eta_\mu(t,\boldsymbol{x}) := \big\langle \mu(\boldsymbol{U}^0(\boldsymbol{x}-\boldsymbol{v}t);\boldsymbol{v}), 1\big\rangle_{\boldsymbol{v}} \tag{A.6}$$

defines a C^∞-smooth function $t \mapsto \eta_\mu(t,\cdot)$ which maps $\mathbb{R}$ into the set of generalized functions $\mathcal{D}'(\mathbb{R}^d)$. More precisely, we assume that the range of the measurable initial value $\boldsymbol{U}^0$ is a compact subset of $\mathcal{S}$, i.e., $\boldsymbol{U}^0(\mathbb{R}^d) \subset\subset \mathcal{S}$. This implies that $\boldsymbol{U}^0$ is bounded and stays away from $\partial\mathcal{S}$. Then for a given $\phi \in \mathcal{E}(\mathbb{R}^d)$ and $\mu \in \mathcal{K}$,

$$\psi \mapsto \big\langle \mu(\boldsymbol{U}^0(\boldsymbol{x}-\boldsymbol{v}t);\boldsymbol{v}), \psi(\boldsymbol{x})\phi(\boldsymbol{v})\big\rangle_{(\boldsymbol{x},\boldsymbol{v})}, \qquad \psi \in \mathcal{D}(\mathbb{R}^d)$$

defines a distribution in $\boldsymbol{x}$ which we denote $\big\langle \mu(\boldsymbol{U}^0(\boldsymbol{x}-\boldsymbol{v}t);\boldsymbol{v}), \phi(\boldsymbol{v})\big\rangle_{\boldsymbol{v}}$. Choosing $\phi \equiv 1$, we see that for each $t \in \mathbb{R}$ the approximation (A.6) is an element of $\mathcal{D}'(\mathbb{R}^d)$ and the dependence on t is C^∞.

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REFERENCES

[1] A. M. ANILE, M. JUNK, V. ROMANO, AND R. RUSSO, *Cross-validation of numerical schemes for extended hydrodynamical models of semiconductors*, Math. Models Methods Appl. Sci., 10 (2000), pp. 833–861.

[2] M. BÄCKER AND K. DRESSLER, *A kinetic method for strictly nonlinear scalar conservation laws*, Z. Angew. Math. Phys., 42 (1991), pp. 243–256.

[3] F. BOUCHUT, *Construction of BGK models with a family of kinetic entropies for a given system of conservation laws*, J. Statist. Phys., 95 (1999), pp. 113–170.

[4] Y. BRENIER, *Une application de la symetrisation de Steiner aux equations hyperboliques: La methode de transport et ecroulement*, C. R. Acad. Sci. Paris Sér. I Math., 292 (1981), pp. 563–566.

[5] Y. BRENIER, *Resolution d'equations d'evolution quasilineaires en dimension N d'espace a l'aide d'equations lineaires en dimension $N+1$*, J. Differential Equations, 50 (1983), pp. 375–390.

[6] Y. BRENIER, *Averaged multivalued solutions for scalar conservation laws*, SIAM J. Numer. Anal., 21 (1984), pp. 1013–1037.

[7] Y. BRENIER, *Averaged Multivalued Solutions and Time Discretization for Conservation Laws*, Lectures in Appl. Math. 22, AMS, Providence, RI, 1985.

[8] Y. BRENIER, *A kinetic formulation for multi–branch entropy solutions of scalar conservation laws*, Ann. Inst. H. Poincare Anal. Non Lineaire, 15 (1998), pp. 169–190.

[9] Y. BRENIER AND E. GRENIER, *Sticky particles and scalar conservation laws*, SIAM J. Numer. Anal., 35 (1998), pp. 2317–2328.

[10] C. CERCIGNANI, *The Boltzmann Equation and Its Applications*, Springer-Verlag, New York, 1988.

[11] P. N. CHILDS AND K. W. MORTON, *Characteristic Galerkin methods for scalar conservation laws in one dimension*, SIAM J. Numer. Anal., 27 (1990), pp. 553–594.

[12] J.-P. CROISILLE, R. KHANFIR, AND G. CHANTEUR, *Numerical simulation of the MHD equations by a kinetic–type method*, J. Sci. Comput., 10 (1995), pp. 81–92.

[13] S. M. Deshpande, *Kinetic Theory Based New Upwind Methods for Inviscid Compressible Flows*, AIAA paper 86–0275, American Institute of Aeronautics and Astronautics, New York, 1986.

[14] S. M. Deshpande and J. C. Mandal, *Kinetic Flux–Vector Splitting (KFVS) for the Euler Equation*, Report 87 FM 2, Department of Aerospace Engineering, Indian Institute of Science, Bangalore, India, 1987.

[15] S. S. Deshpande, *A Boltzmann-Taylor-Galerkin FEM for compressible Euler equations*, in Proceedings of the 14th International Conference on Numerical Methods in Fluid Dynamics, S. M. Deshpande, S. Desai, and R. Narasimha, eds., Lecture Notes in Phys. 453, Springer-Verlag, New York, Berlin, 1995, pp. 91–95.

[16] S. M. Deshpande and O. Pironneau, *A kinetic Fourier scheme*, C. R. Acad. Sci. Paris Sér. I Math., 321 (1995), pp. 1011–1016.

[17] Y. Giga and T. Miyakawa, *A kinetic construction of global solutions of first order quasilinear equations*, Duke Math. J., 50 (1983), pp. 505–515.

[18] A. Harten, P. D. Lax, and B. van Leer, *On upstream differencing and Godunov-type schemes for hyperbolic conservation laws*, SIAM Rev., 25 (1983), pp. 35–61.

[19] F. James, Y.-J. Peng, and B. Perthame, *Kinetic formulation for chromatography and some other hyperbolic systems*, J. Math. Pures Appl., 74 (1995), pp. 367–385.

[20] M. Junk, *Kinetic Schemes: A New Approach and Applications*, Shaker Verlag, Aachen, Germany, 1997.

[21] M. Junk, *Exponentially Exact Conservation Laws*, preprint 220, AG Technomathematik, Universität Kaiserslautern, Kaiserslautern, Germany, 2000.

[22] S. Kaniel, *A kinetic model for the compressible flow equation*, Indiana Univ. Math. J., 37 (1988), pp. 537–563.

[23] R. J. LeVeque, *Second order accurate of Brenier's time-discrete method for nonlinear systems of conservation laws*, SIAM J. Numer. Anal., 25 (1988), pp. 1–7.

[24] P. L. Lions, B. Perthame, and E. Tadmor, *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Amer. Math. Soc., 7 (1994), pp. 169–191.

[25] R. Natalini, *A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws*, J. Differential Equations, 148 (1998), pp. 292–317.

[26] B. Perthame, *Boltzmann type schemes for gas dynamics and the entropy property*, SIAM J. Numer. Anal., 27 (1990), pp. 1405–1421.

[27] B. Perthame and E. Tadmor, *A kinetic equation with kinetic entropy functions for scalar conservation laws*, Comm. Math. Phys., 136 (1991), pp. 501–517.

[28] D. I. Pullin, *Direct simulation methods for compressible inviscid ideal–gas flow*, J. Comput. Phys., 34 (1980), pp. 231–244.

[29] R. D. Reitz, *One–dimensional compressible gas dynamic calculations using the Boltzmann equation*, J. Comput. Phys., 42 (1981), pp. 108–123.

[30] R. H. Sanders and K. H. Prendergast, *On the origin of the* 3 *kiloparsec arm*, Astrophys. J., 188 (1974), pp. 489-500.

[31] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, Berlin, 1983.

[32] F. Treves, *Basic Linear Partial Differential Equations*, Academic Press, New York, 1975.