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Tutorial *Dynamical Systems: Theory and Numerics*

No return

Chaos 1:

Consider the logistic differential equation

$$\dot{x} = \lambda x(1-x), \qquad x(0) = x_0. \tag{1}$$

- Find the solution of (1).
- Discretize (1) with the explicit Euler method.
- Calculate approximate solutions in the case $x_0 = 1/100$, $\lambda = 3$ with $\Delta t = 0.01$, $\Delta t = 0.1$ and compare the results with the exact solution.

Here is the chaos (=breakdown of predictability: little errors in the initial value lead to large errors in the prediction). It happens if the time step of the Euler method is too big in relation to λ.

- Compare the numerical approximations obtained with $\Delta t = 1$, $\lambda = 3$, and initial values $x_0 = 1/100$ and $\tilde{x}_0 = x_0 + \epsilon$ where $\epsilon = 10^{-10}$ (or smaller). Observe how the difference in the approximations grows!

Chaos 2

Rewrite the Euler iteration $x_{n+1} = \tilde{T}_{\Delta t} x_n$ of the previous exercise ($\Delta t = 1$, $\lambda = 3$) in the form of a quadratic iteration

$$y_{n+1} = y_n^2 + c. \tag{2}$$

Show that the relation $x_n = f(y_n)$ is invertible. Despite this mathematical equivalence, chaos hits again ...

- Take the initial value $x_0 = 1/100$ for the Euler method and $y_0 = f(x_0)$ for the iteration (2). Compare the results x_n and $f^{-1}(y_n)$ obtained with the equivalent iterations. Which of the two results is correct?

please turn over

Chaos 3

In the context of quadratic iterations and chaos, the famous Mandelbrot set should be mentioned. It is based on the iteration

$$z_{n+1} = z_n^2 + c, \qquad c \in \mathbb{C}. \tag{3}$$

To construct the Mandelbrot set, you have to plot all constants c in the complex plane for which the iteration (3) with $z_0 = 0$ does not diverge – that's it. Practically you do the following:

1) Take a $c \in [-2, 0.5] \times [-1.25, 1.25]$ and set $z_0 = 0$.
2) Use (3) to iterate until $|z_n| \geq 2$ (it can be shown that the iteration diverges in that case) or until $n > n_{\max}$ (for example $n_{\max} = 40$).
3) If $n > n_{\max}$ you mark the point c in the complex plane by a little dot.

The assumption in this implementation is that $n > n_{\max}$ implies convergence of the sequence which is in general not true. Especially near the boundary of the Mandelbrot set you can find arbitrarily close parameters c which lead to either (extremely slow) convergent or (extremely slow) divergent sequences. To distinguish these cases, $n_{\max}$ must be very large which makes the program very slow (to determine one point of the set, the iteration (3) has to be used very often). On the other hand, the structure of the boundary which appears for large $n_{\max}$ (on big computers) is amazingly rich! Many books on chaos have nice pictures of the Mandelbrot set (e.g. Brian Kaye, *Chaos and Complexity*, VCH, 1993; in KL: PHY 385/081, Handapparat GLTS).

Below is a crude example which you can refine by taking more points c into account. You can also focus on a special part of the boundary by choosing c only from a small rectangle centered at a boundary point. You can also … [end of Exercises].