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WS 1999/00

Tutorial *Dynamical Systems: Theory and Numerics*

Return: 12:00, Friday, 19.11. metal box: Junk/Jungemann (48, ground floor)

Exercise 8:

Let $A > 0$, $n \in \mathbb{N}$ and $x_\circ \in \mathbb{R}^n$. Assume $f : [0, A] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfies the uniform Lipschitz condition $\|f(t,x) - f(t,y)\| \leq L\|x - y\|$ for all $x, y \in \mathbb{R}^n$ and $t \in [0, A]$. Using steps (a) to (c) below, show that the operator T defined by

$$(Tx)(t) := x_\circ + \int_0^t f(\tau, x(\tau))\, d\tau \qquad t \in [0, A]$$

has a unique fixed point $x \in C^0([0, A])$.

a) Show that $C^0([0, A])$ with the norm $\|x\|_L := \max\{\|x(t)\|e^{-2Lt} : t \in [0, A]\}$ is a Banach space.

b) Show that T is a contraction mapping in the $\|\cdot\|_L$–norm.

c) Apply Banach's fixed point theorem to complete the argument.

Exercise 9:

Using the notation from Exercise 8, let $A = 1$, $n = 1$, $x_\circ = 0$ and, for $t \in [0, 1]$,

$$f(t,x) = \begin{cases} 2t & x \leq 0 \\ 2t - 4x/t & 0 < x \leq t^2 \\ -2t & t^2 < x \end{cases}$$

a) Show that f is continuous but not Lipschitz continuous.

b) Show that Picard's iteration $x_{n+1} = Tx_n$ with $x_0 = 0$ does not converge.

c) Show that the limits of the convergent subsequences are not solutions of $\dot{x} = f(t,x)$ with $x(0) = 0$.

d) Find two solutions of $\dot{x} = f(t,x)$ with $x(0) = 0$.

Exercise 10

Today the population of an island is 10.000. The difference between birthrate and deathrate is 0.02 people/year. Each year 1.000 people immigrate. What will be the population of this island ten years from now? When will the population reach 70.000 people? Which numbers will you get if the island's government limits the number of immigrants to 500 per year?