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Tutorial *Dynamical Systems: Theory and Numerics*

Return: 12:00, Friday, 03.12. metal box: Junk/Jungemann (48, ground floor)

Exercise 14:

Consider the initial value problem $\dot{x} = f(t,x)$ with $x(0) = x_\circ \in \mathbb{R}$, where $f : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ is continuous in t and locally Lipschitz continuous in x (i.e. for any $R, T > 0$ there exists $L(T,R) > 0$ such that $|f(t,x) - f(t,y)| \leq L(T,R)|x-y|$ whenever $0 \leq t \leq T$ and $|x|, |y| \leq R$). Moreover assume an estimate of the form

$$|f(t,x)| \leq \alpha(t)|x| + \beta(t)$$

with $\alpha, \beta \in C^0(\mathbb{R}^+)$ and $\alpha, \beta \geq 0$. Prove existence and uniqueness of a solution on the time interval $[0, \infty)$.

Exercise 15:

Consider $\dot{x} = a(x)$ with $a(x) = \alpha$, respectively $a(x) = \alpha x$ where $x \in \mathbb{R}$ and $\alpha > 0$.

- Calculate the phase flow $T_t = T_{t,0}$ for the two cases.
- Check that $T_{-t} = (T_t)^{-1}$.
- Sketch the solution curves $t \mapsto T_t x$ in the (t,x)–plane.

Exercise 16:

Let $a \in C^1(\mathbb{R})$ have a bounded derivative and let $T_t : \mathbb{R} \mapsto \mathbb{R}$ be the phase flow related to $\dot{x} = a(x)$. We consider the problem to find $u \in C^1(\mathbb{R}^+ \times \mathbb{R})$ which satisfies the equation

$$\frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} = 0, \qquad u(0,x) = u_0(x) \tag{1}$$

where $u_0 \in C^1(\mathbb{R})$ is given.

- Show that if u solves (1), then u is constant along the curves $t \mapsto T_t x$, i.e. $u(t, T_t x) = const$ for all $t \geq 0$.
- Conclude that u can be written as $u(t,x) = u_0(T_{-t}x)$.
- Use this result to show that there is at most one solution of (1).
- Show that $(t,x) \mapsto u_0(T_{-t}x)$ satisfies problem (1). (Check first that the function is differentiable, calculate the partial derivatives and plug them into the left hand side of (1). You obtain an expression of the form $y(t)u_0'(T_{-t}x)$. To show that $y(t) \equiv 0$, consider $\dot{y}$ and $y(0)$.)
- Using a from Exercise 15 with $\alpha = 1$ and $u_0(x) = \exp(-x^2)$, sketch the solution $x \mapsto u(1,x)$ for the two cases.