APPENDIX B

Generalized functions

B.1. Generalizing derivatives

In the theory of generalized functions it is possible to extend the domain of definition of differential operators from classically differentiable functions to a larger class in a natural way. Of course, in the theory of differential equations, this is of especial importance. To give an example, let us consider the function $f(x) = |x|$. What is a natural candidate for $f'(x)$? Clearly, for negative x, f is classically differentiable and $f'(x) = -1$. Similarly, $f'(x) = 1$ for positive x. At this point, we already see that any natural derivative of f cannot be continuous at $x = 0$ since its value should be one to the right and minus one to the left. Therefore, the object f' must be an element of some larger function space then $C^0(\mathbb{R})$. Due to the behavior at $x = 0$, it is convenient to look for $f' \in \mathbb{L}^1_{loc}(\mathbb{R})$ (that means f' is an equivalence class of functions which are integrable over any compact subset of $\mathbb{R}$ and which differ from each other at most on a set of Lebesgue measure zero). Indeed, the difference quotients

$$\left.\frac{f(x+h) - f(x)}{h}\right|_{x=0} = \frac{|h|}{h} = \operatorname{sign} h$$

oscillate between plus and minus one depending on the sign of h. Consequently, there is no natural value for the derivative of f at the single point $x = 0$, but in $\mathbb{L}^1_{loc}(\mathbb{R})$, the value of a single point has no relevance. Hence, we can select for example

$$g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

and define $f' \in \mathbb{L}^1_{loc}(\mathbb{R})$ as the equivalence class of functions which contains g. A nice property of this generalized derivative is that the integration by parts formula is still correct. If we take a smooth function $\phi \in C^\infty(\mathbb{R})$ which is compactly supported (so that $f'\phi$ is integrable), we find

$$\int_{\mathbb{R}} f'\phi \, dx = -\int_{-\infty}^{0} \phi \, dx + \int_{0}^{\infty} \phi \, dx$$

and after applying integration by parts to both integrals on the right

$$\int_{\mathbb{R}} f'\phi \, dx = \int_{-\infty}^{0} x\phi' \, dx - \int_{0}^{\infty} x\phi' \, dx = -\int_{\mathbb{R}} |x|\phi' \, dx = -\int_{\mathbb{R}} f\phi' \, dx.$$

In the same way, a natural, generalized derivative of any continuous, piecewise differentiable function can be defined as an element of $\mathbb{L}^1_{loc}(\mathbb{R})$ and the integration by parts formula remains valid.

In the next step, one can ask for the derivative of a function with a jump (say g defined above). Again, we find classically $g' = 0$ for $x < 0$ and $g' = 0$ for $x > 0$. In $x = 0$, the difference quotient can now attain any positive number. The same approach as before would then imply $g' = 0$ in $\mathbb{L}^1_{loc}$ (i.e. g' is the equivalence class containing the zero function). Then, however, any information about the singular behavior of the difference quotient at $x = 0$ is lost. For the same reason, the integration by parts formula does not hold with this definition since

$$\int_{\mathbb{R}} g\phi' \, dx = -\int_{-\infty}^{0} \phi' \, dx + \int_{0}^{\infty} \phi' \, dx = -2\phi(0)$$

is in general different from

$$\int_{\mathbb{R}} 0 \cdot \phi \, dx = 0.$$

Hence, if we require that for the generalized derivative the integration by parts formula should hold, we have to find a function $h \in \mathbb{L}^1_{loc}(\mathbb{R})$ (our candidate for g') such that

$$\int_{\mathbb{R}} h\phi \, dx = -\int_{\mathbb{R}} g\phi' \, dx = 2\phi(0)$$

for any $\phi \in C_0^\infty(\mathbb{R})$. It is easy to see that no such function exists in $\mathbb{L}^1_{loc}(\mathbb{R})$ (because it must be zero almost everywhere in $\mathbb{R}\backslash\{0\}$). Consequently, one either has to give up the integration by parts formula, or the class of functions has to be more general than $\mathbb{L}^1_{loc}$. The latter implies, on the other hand, that also the notion of the integral has to be generalized such that it makes sense for non–$\mathbb{L}^1_{loc}$ functions.

In the theory of generalized functions, the integral

$$\int_{\mathbb{R}} g\phi \, dx$$

with $g \in \mathbb{L}^1_{loc}(\mathbb{R})$ is considered as a linear functional

$$\phi \mapsto \int_{\mathbb{R}} g\phi \, dx = I_g(\phi), \qquad \phi \in C_0^\infty(\mathbb{R})$$

acting on smooth functions ϕ. Since $g \in \mathbb{L}^1_{loc}(\mathbb{R})$ is uniquely determined if we know $I_g(\phi)$ for a sufficiently large class of functions ϕ, this interpretation is very natural (since no information about g is lost when going over to I_g). In our previous example, the derivative of f (or in the new interpretation I_f) was naturally given by g (resp. I_g) and the integration by parts formula

$$I_g(\phi) = -I_f(\phi'), \qquad \forall \phi \in C_0^\infty(\mathbb{R}) \tag{B.1}$$

was valid. Therefore, it was natural, to call $g = f'$ (resp. $I_g = I'_f$) and (B.1) can be written as

$$I'_f(\phi) = -I_f(\phi'), \qquad \forall \phi \in C_0^\infty(\mathbb{R}).$$

In the second example, we have

$$I_g(\phi') = -2\phi(0), \qquad \forall \phi \in C_0^\infty(\mathbb{R})$$

which is still a linear functional acting on ϕ (given essentially by $\phi \mapsto \Lambda(\phi) = 2\phi(0)$), but it can no longer be written in the form I_h with $h \in \mathbb{L}^1_{loc}(\mathbb{R})$. However, the integration by parts formula remains valid, if we set $I'_g = \Lambda$ since then

$$I'_g(\phi) = \Lambda(\phi) = 2\phi(0) = -I_g(\phi'), \qquad \forall \phi \in C_0^\infty(\mathbb{R}).$$

Consequently, by identifying functions $F \in \mathbb{L}^1_{loc}(\mathbb{R})$ with the functionals I_F acting on so called *test functions* $\phi \in C_0^\infty(\mathbb{R})$, we can define derivatives I'_F such that the integration by parts formula holds by setting

$$I'_F(\phi) := -I_F(\phi'), \qquad \forall \phi \in C_0^\infty(\mathbb{R})$$

(the left hand side being clearly a well defined linear functional on the test functions). The derivative I'_F really generalizes the classical derivative. Indeed, if $F \in C^1(\mathbb{R})$ (which is a subspace of $\mathbb{L}^1_{loc}(\mathbb{R})$), we get

$$I'_F(\phi) = -I_F(\phi') = -\int_{\mathbb{R}} F\phi'\,dx = -\,F\phi|_{-\infty}^{\infty} + \int_{\mathbb{R}} F'\phi\,dx = I_{F'}(\phi)$$

i.e. $I'_F = I_{F'}$. Since I_F and $I_{F'}$ can be identified with the functions F and F', we find that the generalized derivative of F is nothing but the classical derivative F'.

It turns out that the identification of $F \in \mathbb{L}^1_{loc}(\mathbb{R})$ with the linear functional I_F is extremely powerful because it can be used to generalize a lot of concepts known for classical functions to a much wider class.

B.2. Aspects from the theory of generalized functions

Here we will briefly introduce some of the concepts which are important in the treatment of fundamental solutions and Green's functions. A more detailed investigation is given in [**11, 14, 15**]. We start with a more precise definition of the test function space and the generalized functions. In the following, Ω denotes an open set in $\mathbb{R}^n$.

Definition B.1. *A function ϕ defined on Ω is called a test function if $\phi \in C^\infty(\Omega)$ and there is a compact set $K \subset \Omega$ such that the support of ϕ lies in K. The set of all test functions on Ω is denoted by $\mathscr{D}(\Omega) = C_0^\infty(\Omega)$.*

Obviously, $\mathscr{D}(\Omega)$ is a linear space. To do analysis, we need a notion of convergence. It is possible to define a topology on $\mathscr{D}(\Omega)$ but for most purposes in PDEs this is not necessary; only a definition for the convergence of sequences is required.

Definition B.2. *Let ϕ_k, $k \in \mathbb{N}$ and ϕ be elements of $\mathscr{D}(\Omega)$. We say that ϕ_k converges to ϕ in $\mathscr{D}(\Omega)$, if there is a compact subset K of Ω such that the supports of all the ϕ_k (and of ϕ) lie in K and, moreover, ϕ_k and derivatives of ϕ_k of arbitrary order converge uniformly to those of ϕ.*

As motivated in the previous section, it is possible to view classical functions as linear functionals on the space of test functions which gives rise to

Definition B.3. *A distribution or generalized function is a linear mapping $\phi \mapsto \Lambda(\phi)$ from $\mathscr{D}(\Omega)$ to $\mathbb{R}$, which is continuous in the following sense: If $\phi_k \to \phi$ in $\mathscr{D}(\Omega)$, then $\Lambda(\phi_k) \to \Lambda(\phi)$. The set of all distributions is called $\mathscr{D}'(\Omega)$. A sequence Λ_k in $\mathscr{D}'(\Omega)$ converges to $\Lambda \in \mathscr{D}'(\Omega)$ if $\Lambda_k(\phi) \to \Lambda(\phi)$ for every $\phi \in \mathscr{D}(\Omega)$.*

It is customary, to write $\langle \Lambda, \phi \rangle$ instead of $\Lambda(\phi)$. Also, $\langle I_f, \phi \rangle$ is denoted $\langle f, \phi \rangle$ due to the possible identification of f and I_f for $f \in \mathbb{L}^1_{loc}(\Omega)$. If the test function ϕ depends on an additional parameter $\phi(x, y)$, it is sometimes not clear without further information on which variable the dual product $\langle \cdot, \cdot \rangle$ acts. Therefore we also write $\langle \Lambda(x), \phi(x, y) \rangle_x$ to indicate that Λ acts on x and not on y. Note, however, that Λ is not an ordinary function and that $\Lambda(x)$ therefore does not refer to the value of Λ at a point x (which is meaningless). It just clarifies how Λ is applied to test functions.

Let us now give a few examples how properties of classical functions can be extended to generalized functions by duality.

Definition B.4. *Let $\Lambda \in \mathscr{D}'(\Omega)$. Then, the derivative of Λ with respect to x_j is defined as*

$$\left\langle \frac{\partial \Lambda}{\partial x_j}, \phi \right\rangle := - \left\langle \Lambda, \frac{\partial \phi}{\partial x_j} \right\rangle, \qquad \forall \phi \in \mathscr{D}(\Omega).$$

Higher derivatives are obtained recursively. For a given multi–index α, we have

$$\langle \nabla^\alpha \Lambda, \phi \rangle = (-1)^{|\alpha|} \langle \Lambda, \nabla^\alpha \phi \rangle .$$

Motivated by the behavior of the integral under a linear transformation of variables $U(x) = Ax + b$ we get

Definition B.5. *Let $U(x) = Ax + b$ be a non singular linear transformation in $\mathbb{R}^n$, and let $U^{-1}(y) = A^{-1}(y - b)$ be the inverse transformation. Then, we set for any $\Lambda \in \mathscr{D}'(\mathbb{R}^n)$*

$$\langle U\Lambda, \phi \rangle := |\det A| \langle \Lambda, \phi \circ U \rangle , \qquad \forall \phi \in \mathscr{D}(\mathbb{R}^n).$$

Another way of denoting the linear transformation is

$$\left\langle \Lambda(U^{-1}(y)), \phi(y) \right\rangle_y = |\det A| \langle \Lambda(x), \phi(Ax + b) \rangle_x .$$

As example, let us mention transformations of Dirac's delta distribution $\delta \in \mathscr{D}'(\mathbb{R}^n)$ defined by

$$\langle \delta, \phi \rangle := \phi(0).$$

Taking any vector $x_0 \in \mathbb{R}^n$, we get the translated delta distribution by

$$\langle \delta(y - x_0), \phi(y) \rangle = \langle \delta(x), \phi(x + x_0) \rangle = \phi(x_0)$$

and for some constant $a \neq 0$

$$\langle \delta(ay), \phi(y) \rangle = \frac{1}{|a|^n} \langle \delta(x), \phi(x/a) \rangle = \frac{\phi(0)}{|a|^n}.$$

The product of a distribution with a smooth function is also given by duality.

Definition B.6. *Let $\psi \in C^\infty(\Omega)$ and $\Lambda \in \mathscr{D}'(\Omega)$. Then $\psi\Lambda$ is defined by*

$$\langle \psi\Lambda, \phi \rangle := \langle \Lambda, \psi\phi \rangle, \qquad \forall \phi \in \mathscr{D}(\Omega).$$

Note that if $\phi \in \mathscr{D}(\Omega)$ then also $\psi\phi$ is a compactly supported C^∞ function on Ω so that the right hand side is well defined.

While products between smooth functions and distributions can be defined, a generalization of the point wise product of functions is not possible in a natural way. More precisely, if Λ_1 and Λ_2 are two distributions, we cannot assign a distribution $\Pi(\Lambda_1, \Lambda_2)$ in such a way that it amounts to the usual product for regular distributions (those which correspond to an $\mathbb{L}^1_{loc}$ function) and is at the same time continuous in both arguments. To see this, just consider the regular distribution $\Lambda_k(x) = \sin(kx)$ in $\mathscr{D}'(\mathbb{R})$. Then $\Pi(\Lambda_k, \Lambda_k) = \sin^2(kx)$, but although $\Lambda_k \to 0$ in $\mathscr{D}'$, we get $\Pi(\Lambda_k, \Lambda_k) \to \frac{1}{2}$. Indeed, for any test function ϕ, we have

$$\langle \Lambda_k, \phi \rangle = \int_{\mathbb{R}} \sin(kx)\phi(x)\, dx = -\frac{1}{k} \int_{\mathbb{R}} \cos(kx)\phi'(x)\, dx$$

so that

$$| \langle \Lambda_k, \phi \rangle | \xrightarrow[k \to \infty]{} 0 \qquad \forall \phi \in \mathscr{D}(\mathbb{R})$$

which means $\Lambda_k \to 0$ in $\mathscr{D}'(\mathbb{R})$. Similarly,

$$\begin{aligned} \langle \Pi(\Lambda_k, \Lambda_k), \phi \rangle &= \int_{\mathbb{R}} \sin^2(kx)\phi(x)\, dx = \frac{1}{2} \int_{\mathbb{R}} (1 - \cos(2kx))\phi(x)\, dx \\ &= \frac{1}{2} \int_{\mathbb{R}} \phi(x)\, dx + \frac{1}{4k} \int_{\mathbb{R}} \sin(2kx)\phi'(x)\, dx \xrightarrow[k \to \infty]{} \frac{1}{2} \int_{\mathbb{R}} \phi(x)\, dx = \left\langle \frac{1}{2}, \phi \right\rangle. \end{aligned}$$

Although the point wise product cannot be generalized, an extension of tensor products is easily possible. That is, if $\Lambda_p \in \mathscr{D}'(\mathbb{R}^p)$ and $\Lambda_q \in \mathscr{D}'(\mathbb{R}^q)$, then $\Lambda_p(x)\Lambda_q(y)$ can be defined as a distribution on $\mathscr{D}'(\mathbb{R}^{p+q})$.

Definition B.7. *Let $\Lambda_p \in \mathscr{D}'(\mathbb{R}^p)$ and $\Lambda_q \in \mathscr{D}'(\mathbb{R}^q)$. Then the direct product $\Lambda_p \otimes \Lambda_q(x, y) = \Lambda_p(x)\Lambda_q(y)$ is the distribution on $\mathbb{R}^{p+q}$ given by*

$$\langle \Lambda_p \otimes \Lambda_q, \phi \rangle = \left\langle \Lambda_p(x), \langle \Lambda_q(y), \phi(x, y) \rangle_y \right\rangle_x, \qquad \forall \phi \in \mathscr{D}(\mathbb{R}^{p+q}).$$

This definition needs some justifications: The function $\psi(x)$ which is obtained when applying the distribution Λ_q to $\phi(x, y)$ is compactly supported and C^∞ smooth. Thus, the right hand side in the above relation is well defined. It remains to be shown that $\Lambda_p \otimes \Lambda_q$ is continuous on $\mathscr{D}(\mathbb{R}^{p+q})$ (see for example [**14**]).

As a simple example of a direct product, we note that $\delta(x)\delta(y) = \delta(x,y)$. If $\phi(x,y)$ itself is of product form $\phi(x,y) = \phi_p(x)\phi_q(y)$, we obtain

$$\langle \Lambda_p \otimes \Lambda_q, \phi \rangle = \langle \Lambda_p, \phi_p \rangle \langle \Lambda_q, \phi_q \rangle .$$

In [**14**], it is shown that linear combinations of products $\phi_p(x)\phi_q(y)$ with $\phi_p \in \mathscr{D}(\mathbb{R}^p)$ and $\phi_q \in \mathscr{D}(\mathbb{R}^q)$ are dense in $\mathscr{D}(\mathbb{R}^{p+q})$. This fact can be used to show properties of the direct product by verifying them only on the restricted set of test functions of product form. One immediate conclusion is that in the definition we can evaluate Λ_p and Λ_q in the opposite order, i.e. we also have

$$\langle \Lambda_p \otimes \Lambda_q, \phi \rangle = \left\langle \Lambda_q(y), \langle \Lambda_p(x), \phi(x,y) \rangle_x \right\rangle_y$$

which we express by the suggestive notation

$$\Lambda_p(x)\Lambda_q(y) = \Lambda_q(y)\Lambda_p(x).$$

Another property is the associative law

$$\Lambda_p(x)(\Lambda_q(y)\Lambda_r(z)) = (\Lambda_p(x)\Lambda_q(y))\Lambda_r(z).$$

Finally, we turn to the convolution of two distributions. To motivate the definition, let us consider the case of two classical functions f, g which decay rapidly at infinity. Then

$$\begin{aligned} \langle f * g, \phi \rangle &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y) g(y)\, dy \right) \phi(x)\, dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(y) \phi(x+y)\, dx\, dy. \end{aligned}$$

While the product $f(x)g(y)$ can be generalized as direct product of two distributions, the problem is that $\phi(x+y)$ is not a test function out of $\mathscr{D}(\mathbb{R}^{2n})$ since it is not compactly supported (unless $\phi \equiv 0$). Indeed, it turns out that the convolution of arbitrary distributions cannot be defined in a rational manner. There are, however, special cases where a meaning can be given to $\langle \Lambda_1(x)\Lambda_2(y), \phi(x+y) \rangle_{(x,y)}$. In the simplest such case, the support of $\phi(x+y)$ has a compact intersection with the support of $f(x)g(y)$. (In this case, we may replace $\phi(x+y)$ by any test function which agrees with $\phi(x+y)$ in a neighborhood of $\operatorname{supp}(\Lambda_1(x)\Lambda_2(y))$.) To make this idea more precise, we have to clarify the notion *support of a distribution*.

Although generalized functions cannot be evaluated at points, they can be restricted to open sets. If G is an open subset of Ω, then $\mathscr{D}(G)$ is naturally a subset of $\mathscr{D}(\Omega)$. Consequently, we shall say $\Lambda \in \mathscr{D}'(\Omega)$ vanishes on G if $\langle \Lambda, \phi \rangle = 0$ for all $\phi \in \mathscr{D}(G)$. Two distributions are said to be equal on G if their difference vanishes on G.

It can be shown (by a partition of unity argument) that if Λ vanishes locally around every point of G then Λ vanishes on G. Now, the complement of the union of all open subsets of Ω on which Λ vanishes is a natural candidate for the support of Λ. For example, the support of the delta distribution is the set $\{0\}$. Although the δ cannot be evaluated at points, it makes sense to say that it vanishes except at the origin.

Let Λ be compactly supported on a set $K \subset \Omega$. We then take a test function $\phi_K \in \mathscr{D}(\Omega)$ which is equal to one on a neighborhood of K and find for any $\phi \in \mathscr{D}(\Omega)$

$$\langle \Lambda, \phi \rangle = \langle \Lambda, \phi\phi_K + (1 - \phi_K)\phi \rangle = \langle \phi_K \Lambda, \phi \rangle$$

since $(1 - \phi_K)\phi \in \mathscr{D}(\Omega \backslash K)$. Thus, $\Lambda = \phi_K \Lambda$ and we can even define the dual product with non–compactly supported test functions $\eta \in C^\infty(\Omega)$ by setting

$$\langle \Lambda, \eta \rangle := \langle \Lambda, \phi_K \eta \rangle \, .$$

To show that this definition is independent of the choice of ϕ_K is simple. If $\tilde{\phi}_K$ is another function with the same properties as ϕ_K, then

$$\langle \Lambda, \phi_K \eta \rangle = \left\langle \tilde{\phi}_K \Lambda, \phi_K \eta \right\rangle = \left\langle \Lambda, \tilde{\phi}_K \phi_K \eta \right\rangle = \left\langle \phi_K \Lambda, \tilde{\phi}_K \eta \right\rangle = \left\langle \Lambda, \tilde{\phi}_K \eta \right\rangle .$$

Coming back to the definition of the convolution, we see that if , for example , Λ_1 is compactly supported, we can replace Λ_1 by $\phi_K \Lambda_1$ with a suitable function $\phi_K \in \mathscr{D}(\mathbb{R}^n)$. Then,

$$\langle \Lambda_1 * \Lambda_2, \phi \rangle := \langle \Lambda_1(x)\Lambda_2(y), \phi(x+y) \rangle = \langle \Lambda_1(x)\Lambda_2(y), \phi_K(x)\phi(x+y) \rangle$$

and now $(x, y) \mapsto \phi_K(x)\phi(x+y)$ is really contained in $\mathscr{D}(\mathbb{R}^{2n})$ so that the convolution is well defined.

As example, we consider the convolution of a distribution Λ with the δ distribution (which is compactly supported)

$$\langle \delta * \Lambda, \phi \rangle = \langle \delta(x)\Lambda(y), \phi(x+y) \rangle = \langle \Lambda(y), \langle \delta(x), \phi(x+y) \rangle_x \rangle_y = \langle \Lambda(y), \phi(y) \rangle_y$$

so that $\delta * \Lambda = \Lambda$. If we take the convolution of a distribution Λ with a test function ψ, we find

$$\begin{aligned}
\langle \Lambda * \psi, \phi \rangle &= \left\langle \Lambda(x), \langle \psi(y), \phi(x+y) \rangle_y \right\rangle_x \\
&= \left\langle \Lambda(x), \langle \psi(y-x), \phi(y) \rangle_y \right\rangle_x = \langle \langle \Lambda(x), \psi(x-y) \rangle_x , \phi(y) \rangle_y
\end{aligned}$$

so that $\Lambda * \psi$ is represented by the smooth function $y \mapsto \langle \Lambda(x), \psi(y-x) \rangle_x$. Finally, the derivative of a convolution is

$$\begin{aligned}
\left\langle \frac{\partial}{\partial x_j}(\Lambda_1 * \Lambda_2), \phi \right\rangle &= - \left\langle \Lambda_1(x)\Lambda_2(y), \frac{\partial \phi}{\partial x_j}(x+y) \right\rangle \\
&= - \left\langle \Lambda_2(y), \left\langle \Lambda_1(x), \frac{\partial \phi}{\partial x_j}(x+y) \right\rangle_x \right\rangle_y \\
&= - \left\langle \Lambda_2(y), \left\langle \frac{\partial \Lambda_1}{\partial x_j}(x), \phi \right\rangle_x \right\rangle_y = \left\langle \frac{\partial \Lambda_1}{\partial x_j} * \Lambda_2, \phi \right\rangle
\end{aligned}$$

Similarly, we can show

$$\frac{\partial}{\partial x_j}(\Lambda_1 * \Lambda_2) = \frac{\partial \Lambda_1}{\partial x_j} * \Lambda_2 = \Lambda_1 * \frac{\partial \Lambda_2}{\partial x_j}$$

so that by induction for any multi–index α

$$\nabla^\alpha(\Lambda_1 * \Lambda_2) = (\nabla^\alpha \Lambda_1) * \Lambda_2 = \Lambda_1 * (\nabla^\alpha \Lambda_2).$$

One can also show a result on the support of the convolution $\Lambda_1 * \Lambda_2$. It is contained in the sum of the supports of Λ_1 and Λ_2

$$\text{supp}\,(\Lambda_1 * \Lambda_2) \subset \text{supp}\,\Lambda_1 + \text{supp}\,\Lambda_2.$$

For the (easy) proof we refer to [**15**].

Finally, we want to show that convolution can be used to prove that any $\Lambda \in \mathscr{D}'(\Omega)$ can be approximated by test functions in the sense of distributions. The idea is to choose cut–off functions $\chi_k \in \mathscr{D}(\Omega)$ which are equal to one on large subsets Ω_k of Ω and drop to zero only close to the boundary of Ω. For

$$\Omega_k := \{\, x \in \Omega \,:\, \text{dist}(x, \partial\Omega) > \frac{1}{k}, \quad |x| < k\}$$

it is easy to see that $\chi_k \Lambda \to \Lambda$ in $\mathscr{D}'(\Omega)$ (since any fixed test function will eventually be supported in Ω_k). Since $\chi_k \Lambda \in \mathscr{D}'(\mathbb{R}^n)$ has compact support it suffices to consider only such distributions.

Choosing a mollifier $\phi \in \mathscr{D}(B_1(0))$ which is nonnegative and symmetric ($\phi(x) = \phi(-x)$) and satisfies $\langle 1, \phi\rangle = 1$, we set up

$$\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi\left(\frac{x}{\epsilon}\right).$$

As we have seen, $\Lambda * \phi_\epsilon \in \mathscr{D}(\mathbb{R}^n)$. Moreover, for any $\psi \in \mathscr{D}(\mathbb{R}^n)$

$$\begin{aligned}
\langle \Lambda * \phi_\epsilon, \psi\rangle &= \Big\langle \langle \Lambda(y), \phi_\epsilon(x-y)\rangle_y\,, \psi(x)\Big\rangle_x \\
&= \Big\langle 1, \langle \Lambda(y), \phi_\epsilon(x-y)\psi(x)\rangle_y\Big\rangle_x \\
&= \langle 1 \cdot \Lambda(y), \phi_\epsilon(x-y)\psi(x)\rangle_{(x,y)} \\
&= \langle \Lambda(y), \langle 1, \phi_\epsilon(x-y)\psi(x)\rangle_x\rangle_y \\
&= \langle \Lambda(y), \phi_\epsilon * \psi(y)\rangle_y
\end{aligned}$$

(Note that in each step we have used definitions and results from above.) For smooth functions we can classically show that $\phi_\epsilon * \psi \to \psi$ in $\mathscr{D}(\mathbb{R}^n)$ for $\epsilon \to 0$. Thus

$$\langle \Lambda * \phi_\epsilon, \psi\rangle \xrightarrow[\epsilon \to 0]{} \langle \Lambda, \psi\rangle$$

so that $\Lambda * \phi_\epsilon \to \Lambda$ in $\mathscr{D}'(\mathbb{R}^n)$. Summarizing the result, we get:

Lemma B.8. *Let Ω and Ω' be open subsets of $\mathbb{R}^n$ with $\bar{\Omega}' \subset \Omega$ and let $\phi \in \mathscr{D}(B_1(0))$ be a test function which satisfies $\phi \geq 0$ and $\langle 1, \phi \rangle = 1$. With $\phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon^n$ we have*

$$\Lambda * \phi_\epsilon|_{\Omega'} \xrightarrow[\epsilon\to 0]{\mathscr{D}'(\Omega')} \Lambda|_{\Omega'} .$$

Note that $\Lambda * \phi_\epsilon|_{\Omega'} \in C^\infty(\bar{\Omega}')$ for ϵ small enough.

B.3. A simple PDE

Theorem B.9. *Let $\Omega \subset \mathbb{R}^n$ be open and connected and let $m \in \mathbb{N}$. If $u \in \mathscr{D}'(\Omega)$ satisfies $\nabla^\alpha u = 0$ for all $|\alpha| = m$ then u is a polynomial of degree less than m.*

We start with a Lemma for the case $m = 1$.

Lemma B.10. *Let $\Omega \subset \mathbb{R}^n$ be open and connected. If $u \in \mathscr{D}'(\Omega)$ satisfies $\nabla u = 0$ then u is a constant function on Ω.*

Proof: Let $B \subset \Omega$ be a ball with $\bar{B} \subset \Omega$. Using a mollifier ϕ_ϵ as in Lemma B.8, we find

$$u * \phi_\epsilon \in C^\infty(\bar{B}).$$

Using the fact that derivatives can be put to each partner in the convolution, we find for $x \in B$

$$\nabla(u * \phi_\epsilon)(x) = ((\nabla u) * \phi_\epsilon)(x) = 0$$

so that $u * \phi_\epsilon$ is constant on B, say equal to c_ϵ. If $\psi_0 \in \mathscr{D}(B)$ is a test function which is normalized in the sense $\langle 1, \psi_0 \rangle = 1$, we have

$$c_\epsilon = \langle u * \phi_\epsilon, \psi_0 \rangle \xrightarrow[\epsilon\to 0]{} \langle u, \psi_0 \rangle .$$

(Here, convergence from Lemma B.8 has been used.) For any other $\psi \in \mathscr{D}(B)$ we get

$$\langle u * \phi_\epsilon - \langle u, \psi_0 \rangle, \psi \rangle = (c_\epsilon - \langle u, \psi_0 \rangle) \langle 1, \psi \rangle \xrightarrow[\epsilon\to 0]{} 0$$

which implies

$$\langle u - \langle u, \psi_0 \rangle, \psi \rangle = 0 \tag{B.2}$$

or, in other words,

$$u|_B = \langle u, \psi_0 \rangle .$$

We remark that the constant $c_B = \langle u, \psi_0 \rangle$ is, in fact, independent of ψ_0 and thus depends only on u and B. In fact, if ψ_1 is another test function which is normalized by $\langle 1, \psi_1 \rangle = 1$ we get from (B.2) $\langle u, \psi_1 \rangle = \langle u, \psi_0 \rangle$.

It is easy to see that the constants c_{B_1} and c_{B_2} of two intersecting balls B_1 and B_2 must be equal by taking $\psi \in \mathscr{D}(B_1 \cap B_2)$ with $\langle 1, \psi \rangle = 1$ and observing

$$c_{B_1} = \langle c_{B_1}, \psi \rangle = \langle u, \psi \rangle = \langle c_{B_2}, \psi \rangle = c_{B_2}.$$

Hence, we can define a mapping $f : \Omega \mapsto \mathbb{R}$ which assigns to $x \in \Omega$ the value c_B where $x \in B$ and B is any ball whose closure is contained in Ω. The function f is even continuous. If we take a sequence $(x_k)_{k\in\mathbb{N}}$ which converges to $x \in \Omega$ then, eventually, the members of the sequence will be in a small ball B around x so that $f(x_n) = c_B = f(x)$ for n sufficiently large. Finally, we consider the set

$$A = \{\, x \in \Omega \,:\, f(x) = f(x_0) \,\}$$

where $x_0 \in \Omega$ is an arbitrary point. Since f is continuous on Ω, A is closed in Ω. On the other hand, for any $x \in A$ we can find a ball B around x on which f is equal to $f(x)$ which is equal to $f(x_0)$ by definition of A. Thus, A is also open and since $x_0 \in A \neq \emptyset$ the connectedness of Ω implies $A = \Omega$ which yields $f = const.$ Using a locally finite partition of unity on Ω based on balls $B \subset \Omega$, we can thus infer that $u = f$. ■

The remainder of the proof of Theorem B.9 is just a simple induction argument.

Proof: (of Theorem B.9)

The case $m = 1$ is treated in the Lemma so we assume the Theorem is correct for $m - 1$. Let $u \in \mathscr{D}'(\Omega)$ with $\nabla^\alpha u = 0$ for all $|\alpha| = m$. Set $v_\beta = \nabla^\beta u$ where $|\beta| = m - 1$ so that $\nabla v_\beta = 0$ for each β which implies $v_\beta = c_\beta$ by the Lemma above. Since

$$\nabla^\beta \left(\frac{1}{\beta!} c_\beta x^\beta \right) = c_\beta$$

we set up

$$\hat{u} = u - \sum_{|\beta|=m-1} \frac{1}{\beta!} c_\beta x^\beta \tag{B.3}$$

so that $\nabla^\beta \hat{u} = 0$ for all $|\beta| = m - 1$. Hence, $\hat{u}$ is a polynomial of degree less than $m - 1$ which concludes the proof due to (B.3). ■