APPENDIX C

Sobolev spaces

The following introduction to Sobolev spaces is not meant to be exhaustive. We just want to present the basic concepts and focus somewhat on the problem how to generalize boundary values.

Definition C.1. *Let $\Omega \subset \mathbb{R}^n$ be open and let $m \in \mathbb{N}_0$, $p \in [1, \infty]$. The Sobolev space $W^{m,p}(\Omega)$ consists of those functions $f \in \mathbb{L}^p(\Omega)$ whose distributional derivatives up to order m are again $\mathbb{L}^p(\Omega)$ functions.*

Of course, $W^{m,p}(\Omega)$ can naturally be interpreted as a subset of $\mathscr{D}'(\Omega)$. However, we can choose a stronger topology which is based on the norm

$$\|f\|_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} \|\nabla^\alpha f\|_{0,p}^p \right)^{\frac{1}{p}}.$$

(Note that $W^{0,p}(\Omega) = \mathbb{L}^p(\Omega)$.) It turns out that the norm topology on $W^{m,p}(\Omega)$ is complete [**1**] so that the Sobolev spaces are Banach spaces. The case $p = 2$ is of special importance because $W^{m,2}(\Omega)$ are even Hilbert spaces with the scalar products

$$\langle u, v \rangle_m = \sum_{|\alpha| \leq m} \int_\Omega \nabla^\alpha u \nabla^\alpha v \, dx.$$

A standard abbreviation of $W^{m,2}(\Omega)$ is $H^m(\Omega)$.

C.1. Embedding theorem

An important property of Sobolev spaces are the embedding relations among them. For a complete list of these relations we refer to [**1**]. Here, we just mention a few:

Theorem C.2. *Let Ω be a domain in $\mathbb{R}^n$ with a Lipschitz boundary and let p satisfy $1 \leq p < \infty$. Under the condition $p < n$ we find*

$$W^{1,p}(\Omega) \hookrightarrow \mathbb{L}^q(\Omega), \qquad p \leq q \leq np/(n-p).$$

If $mp > n$ for some $m \in \mathbb{N}$, then

$$W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$$

where $C_B(\Omega)$ is the space of bounded continuous functions on Ω (endowed with the supremum norm). If Ω is bounded and m is a nonnegative integer then

$$W^{m+1,p}(\Omega) \hookrightarrow W^{m,p}(\Omega)$$

and this embedding is even compact *(i.e. every $\|\cdot\|_{m+1,p}$– bounded sequence has a subsequence which converges in $W^{m,p}(\Omega)$.*

Proof: See [**1**]. ∎

In order to illustrate the compact embedding, let us consider the special case $\Omega = (0,1)$, with $p = 2$ and $m = 1$. More precisely, we will show that

$$H^1(0,1) \hookrightarrow \mathbb{L}^2(0,1)$$

is compact. Our argument is based on the knowledge of orthonormal bases in $\mathbb{L}^2(0,1)$ (see Exercise C.4). With

$$\psi_n(x) = \sqrt{2}\sin(n\pi x) \qquad n \in \mathbb{N}$$

we can represent any $v \in \mathbb{L}^2(0,1)$ by the ($\mathbb{L}^2$–convergent) series

$$v = \sum_{n\in\mathbb{N}} \langle v, \psi_n\rangle_0 \, \psi_n$$

where $\langle\cdot,\cdot\rangle_0$ is the scalar product on $\mathbb{L}^2$. According to Parseval's equality, we have

$$\|v\|^2_{\mathbb{L}^2} = \sum_{n\in\mathbb{N}} |\langle v, \psi_n\rangle_0|^2$$

so that the mapping $v \mapsto (\langle v, \psi_n\rangle_0)_{n\in\mathbb{N}}$ is an isometry from $\mathbb{L}^2(0,1)$ to the space of square summable sequences

$$l^2 = \{\, (c_n)_{n\in\mathbb{N}} \,:\, \sum_{n\in\mathbb{N}} |c_n|^2 < \infty \,\}.$$

which is a Hilbert space with the scalar product $\langle c, d\rangle_{l^2} = \sum_{n\in\mathbb{N}} c_n d_n$. Using the cosine basis

$$\phi_0 \equiv 1, \quad \phi_n(x) = \sqrt{2}\cos(n\pi x) \qquad n \in \mathbb{N}$$

we get with the same argument that

$$(\langle v, \phi_n\rangle_0)_{n\in\mathbb{N}} \in l^2 \qquad \forall v \in \mathbb{L}^2(0,1).$$

If we now take $u \in H^1(0,1) \subset \mathbb{L}^2(0,1)$, we observe

$$n\pi \langle u, \phi_n\rangle_0 = \langle u, \psi_n'\rangle_0 = -\langle u', \psi_n\rangle_0 + |u(x)\psi_n(x)|_0^1 .$$

Since ψ_n vanishes at the end points of the interval we conclude

$$(n\langle u, \phi_n\rangle_0)_{n\in\mathbb{N}} = -\frac{1}{\pi}(\langle u', \psi_n\rangle_0)_{n\in\mathbb{N}} \in l^2$$

Hence, for $u \in H^1(0,1)$ the coefficients of the cosine basis are square summable even if they are multiplied by the increasing sequence $(n)_{n\in\mathbb{N}}$. In other words, the high frequency parts of H^1 functions (i.e. the coefficients corresponding to highly oscillatory basis functions) are much smaller than those of $\mathbb{L}^2$ functions. Introducing the space

$$h^1 = \{ c \in l^2 \, : \, \sum_{n\in\mathbb{N}} n^2 |c_n|^2 < \infty \}$$

we conclude that

$$u \in H^1(0,1) \qquad \Rightarrow \qquad (\langle u, \phi_n \rangle_0)_{n\in\mathbb{N}} \in h^1$$

(Note that h^1 is also a Hilbert space based on the scalar product $\langle c, d \rangle_{h^1} = \sum_{n\in\mathbb{N}} n^2 c_n d_n$.) The compact embedding of $H^1(0,1)$ in $\mathbb{L}^2(0,1)$ now easily follows from the compact embedding $h^1 \hookrightarrow l^2$. (The l^2 elements just have to be interpreted as coefficients for the cosine basis of $\mathbb{L}^2(0,1)$. More precisely, if $(u_m)_{m\in\mathbb{N}}$ is a uniformly bounded sequence in $H^1(0,1)$ then

$$\sum_{n\in\mathbb{N}} n^2 |\langle u_m, \phi_n \rangle_0|^2 = \frac{1}{\pi^2} \sum_{n\in\mathbb{N}} |\langle u_m', \psi_n \rangle_0|^2 = \frac{1}{\pi^2} \|u_m'\|_{\mathbb{L}^2}^2 \leq \frac{1}{\pi^2} \|u_m\|_{H^1}^2$$

so that $(c_n^{(m)})_{n\in\mathbb{N}}(\langle u_m, \phi_n \rangle_0)_{n\in\mathbb{N}}$ gives rise to a uniformly bounded sequence in h^1. Due to compactness, we can extract an l^2 convergent subsequence which corresponds to a sequence in $\mathbb{L}^2(0,1)$ with the desired properties.) To show the compactness of $h^1 \hookrightarrow l^2$ we take a uniformly bounded sequence $(c^{(m)})_{m\in\mathbb{N}} \in h^1$, i.e. $\|c^{(m)}\|_{h^1} \leq C$ for all $m \in \mathbb{N}$. This implies, that each component $c_n^{(m)}$ is a bounded sequence in $\mathbb{R}$. Using a diagonal process, we can therefore find a subsequence $(c^{(m_k)})_{k\in\mathbb{N}}$ which converges in each component n to some $c_n \in \mathbb{R}$. The resulting sequence is again in h^1 since

$$\|c\|_{h^1}^2 = \sum_{n\in\mathbb{N}} n^2 |c_n|^2 = \sum_{n\in\mathbb{N}} \lim_{k\to\infty} n^2 |c_n^{(m_k)}|^2 \leq \liminf_{k\to\infty} \sum_{n\in\mathbb{N}} n^2 |c_n^{(m_k)}|^2 \leq C^2$$

where the discrete version of Fatou's Lemma has been used. To show that the extracted subsequence converges in l^2 we use the observation that for any $d \in h^1$ we can estimate the tail $\mathcal{X}_{n>N} d$ according to

$$\|\mathcal{X}_{n>N} d\|_{l^2} = \left(\sum_{n>N} |d_n|^2 \right)^{\frac{1}{2}} \leq \frac{1}{N} \left(\sum_{n>N} n^2 |d_n|^2 \right)^{\frac{1}{2}} \leq \frac{\|d\|_{h^1}}{N}.$$

($\mathcal{X}_{n>N}$ is a sequence which is equal to one for $n > N$ and zero otherwise. The product $\mathcal{X}_{n>N} d$ refers to a component wise product of two sequences.) To a given $\epsilon > 0$ we then pick N such that $C/N < \epsilon$. After that, the leading parts $\mathcal{X}_{n\leq N} c^{(m_k)}$ can be replaced by $\mathcal{X}_{n\leq N} c$ with an error of at most ϵ if k is sufficiently large because $c_n^{(m_k)} \to c_n$ for each n. Altogether,

$$\|c^{(m_k)} - c\|_{l^2} \leq \|\mathcal{X}_{n\leq N}(c^{(m_k)} - c)\|_{l^2} + \|\mathcal{X}_{n>N} c^{(m_k)}\|_{l^2} + \|\mathcal{X}_{n>N} c\|_{l^2} \leq 3\epsilon.$$

which shows that the subsequence converges in l^2 and thus completes the proof. For more general results in the case of sequence spaces we refer to Exercise C.5.

C.2. Generalized traces

It turns out that for elements of Sobolev spaces with $m \geq 1$ the restriction to the boundary $\partial\Omega$ (the so called *trace*) can be defined in a natural way provided $\partial\Omega$ is sufficiently smooth. One reasons for this is that differentiable functions are dense in the Sobolev space and for these functions the trace is obtained in the usual way. Second, the trace operation turns out to be a bounded linear mapping into certain $\mathbb{L}^q$ spaces on $\partial\Omega$ so that the natural trace for smooth functions can be extended to all elements of the Sobolev space. More precisely, the space $C^m(\bar{\Omega})$ is dense in $W^{m,p}(\Omega)$ if Ω has the so called *segment property* (to each $x \in \partial\Omega$ there exists a neighborhood U_x and a vector $y \neq 0$ such that with $z \in U_x \cap \bar{\Omega}$ the whole segment $z + ty$, $t \in (0,1)$ belongs to Ω; the segment property rules out that Ω lies on both sides on parts of its boundary).

Theorem C.3. *If $\Omega \subset \mathbb{R}^n$ has the segment property, then the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$ for $1 \leq p < \infty$. If $u \in C(\bar{\Omega}) \cap W^{m,p}(\Omega)$ then u can be approximated in $W^{m,p}$ and locally uniformly by $C_0^\infty(\mathbb{R}^n)$-functions.*

Proof: The proof is based on a partition of unity argument. Locally at the boundary, the $W^{m,p}$ function is slightly moved outside the domain (along the vector y provided by the segment property) and afterwards smoothed by mollification. The resulting sequence of functions is easily seen to converge in $W^{m,p}$ due to the continuity of the translation operator (in $\mathbb{L}^p$) and the convergence of mollification (in $\mathbb{L}^p$). For more details, we refer to [**1**]. The part concerning continuous u is contained in the argument but not explicitly mentioned. ∎

Due to Theorem C.3, the trace operation

$$Bf = f|_{\partial\Omega}$$

makes sense at least for a dense subset of $W^{m,p}(\Omega)$. To extend the trace operator B to the whole space $W^{m,p}(\Omega)$ we need some boundedness property with respect to $\|\cdot\|_{m,p}$. Indeed, if we have

$$\|Bf\|_X \leq C\|f\|_{m,p,\Omega} \qquad \forall f \in C_0^\infty(\mathbb{R}^n) \tag{C.1}$$

for some Banach space X then we can define for any $f \in W^{m,p}(\Omega)$

$$Bf := \lim_{n\to\infty} Bf_k \tag{C.2}$$

where $(f_k)_{k\in\mathbb{N}}$ is a sequence in $C_0^\infty(\mathbb{R}^n)$ which converges to f in $W^{m,p}(\Omega)$ (such sequences exist due to Theorem C.3). The limit in (C.2) exists since X is a Banach space and Bf_k is, in fact, a Cauchy sequence

$$\|Bf_k - Bf_l\|_X \leq C\|f_k - f_l\|_{m,p,\Omega}.$$

Also, the definition (C.2) is independent of the chosen sequence. If $(g_k)_{k\in\mathbb{N}}$ is another sequence with the same properties as $(f_k)_{k\in\mathbb{N}}$ then

$$\|Bf_k - Bg_k\|_X \leq C\|f_k - g_k\|_{m,p} \leq C(\|f_k - f\|_{m,p} + \|g_k - f\|_{m,p}) \xrightarrow[n\to\infty]{} 0.$$

The crucial estimate (C.1) can be obtained for $X = \mathbb{L}^q(\partial\Omega)$ with a suitable $q \in [1,\infty)$ if we assume that Ω is a bounded domain with a Lipschitz continuous boundary and if $m \geq 1$. In general, it turns out that the trace bf of a $W^{m,p}$–function f with $m \geq 1$ has less regularity than f itself. A detailed investigation of this issue gives rise to Sobolev spaces of fractional order $X = W^{s,q}(\partial\Omega)$ with $s \in \mathbb{R}^+$. Under suitable assumptions on the smoothness of $\partial\Omega$ one can show, for example, that B maps $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\partial\Omega)$ (see [**12**]). We will not go into these details but content ourselves with the case $X = \mathbb{L}^q(\Omega)$.

The boundary of Ω is assumed to be at least Lipschitz continuous in the following. That means, locally $\partial\Omega$ can be described as the graph of a Lipschitz continuous function with Ω lying on one side of the surface (if $\partial\Omega$ is unbounded, additional uniform bounds are required for example for the Lipschitz constants of the functions describing the surface). For a detailed discussion we refer to [**12**]. The general idea is to reduce the investigations to half space problems by suitable transformations and partition of unity arguments. Since these auxiliary steps are only of technical nature we focus our attention solely on the case

$$\Omega = \{\, x \in \mathbb{R}^n \ : \ x_n > 0 \,\}$$

to exhibit the main arguments. (Observe that Ω is not bounded but $\partial\Omega$ is so simple that the ideas necessary for bounded domains also apply.)

Theorem C.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{0,1}$ domain. Then there exists a uniquely determined continuous linear mapping $B : W^{1,p}(\Omega) \mapsto \mathbb{L}^q(\partial\Omega)$ such that $Bu = u|_{\partial\Omega}$ for all $u \in C(\bar{\Omega}) \cap W^{1,p}(\Omega)$. If $1 \leq p < n$ then $1 \leq q \leq (np-p)/(n-p)$. Otherwise, any $q \in [1,\infty)$ is possible. In particular, B maps $W^{1,p}(\Omega)$ into $\mathbb{L}^p(\partial\Omega)$ and the image is dense.*

Proof: First, we consider the case $1 \leq p < n$ and $q = (np-p)/(n-p)$. For a detailed proof we refer to [**12**] and consider only the case where Ω is a half space. We will show for $u \in C_0^\infty(\mathbb{R}^n)$ that

$$\|Bu\|_{0,q,\partial\Omega} \leq C\|u\|_{1,p,\Omega}.$$

In view of our considerations above, this suffices to prove the existence of the trace operator B on $W^{1,p}(\Omega)$. We first note that

$$q = \frac{n-1}{n-p}p \geq p \geq 1.$$

If $u \in C_0^\infty(\mathbb{R}^n)$, there exists $R > 0$ such that $u(\hat{x}, R) = 0$ for all $\hat{x} \in \mathbb{R}^{n-1}$. Applying the fundamental theorem of calculus, we easily get

$$\text{(C.3)} \qquad |u(\hat{x}, 0)|^q \leq \int_0^R |u|^{q-1} \left| \frac{\partial u}{\partial x_n} \right| (\hat{x}, x_n)\, dx_n.$$

Integrating over $\partial\Omega$ and applying Hölder's inequality, we obtain

$$\|Bu\|_{0,q,\partial\Omega}^q \leq q \|u\|_{0,\frac{np}{n-p},\Omega}^{q-1} \|u\|_{1,p,\Omega}$$

where we have used the relation

$$\frac{q-1}{1-\frac{1}{p}} = \frac{np}{n-p}.$$

For $q = 1 = p$ the norm estimate is already finished. In the other cases we use the Sobolev embedding theorem C.2 which shows that $W^{1,p}(\Omega)$ can be viewed as a subspace of $\mathbb{L}^{\frac{np}{n-p}}(\Omega)$ with

$$\|u\|_{0,\frac{np}{n-p},\Omega} \leq C \|u\|_{1,p,\Omega}$$

so that we can conclude

$$\|Bu\|_{0,q,\partial\Omega} \leq q C^{\frac{q-1}{q}} \|u\|_{1,p,\Omega}.$$

The case $1 \leq p < n$ and $1 \leq q \leq (np - p)/(n - p)$ (now with bounded Ω)is reduced to the previous one with the remark that $\mathbb{L}^{q_2}(\partial\Omega)$ is continuously embedded into $\mathbb{L}^{q_1}(\partial\Omega)$ if $q_1 \leq q_2$ since $\partial\Omega$ is bounded, i.e.

$$\|Bu\|_{0,q_1,\partial\Omega} \leq C \|Bu\|_{0,q_2,\partial\Omega} \qquad \text{for } q_1 \leq q_2.$$

Similarly, $p \geq n$ is reduced to the case above. Indeed, picking any $q \in [1, \infty)$ we can find $\hat{p} \in [1, n)$ such that $q = (n\hat{p} - \hat{p})/(n - \hat{p})$. Consequently, B is a bounded linear operator from $W^{1,\hat{p}}(\Omega)$ to $\mathbb{L}^q(\partial\Omega)$. Since $\partial\Omega$ is bounded and $\hat{p} < p$ we also have

$$\|u\|_{1,\hat{p},\Omega} \leq C \|u\|_{1,p,\Omega}.$$

In both cases we see that B maps $W^{1,p}(\Omega)$ into $\mathbb{L}^p(\partial\Omega)$. To show that the image is dense we consider again the simple (but crucial) case where Ω is the half space. For a given $\phi \in C_0^\infty(\mathbb{R}^{n-1})$ we can easily find a function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $B\phi = \psi$. Indeed, if $\theta \in C_0^\infty(\mathbb{R})$ is a function which is equal to one at the origin, we simply set $\phi(\hat{x}, x_n) = \psi(\hat{x})\theta(x_n)$. Since $\phi|_\Omega \in W^{1,p}(\Omega)$ we find that $C_0^\infty(\mathbb{R}^{n-1})$ is contained in $B(W^{1,p}(\Omega))$ which is therefore dense in $\mathbb{L}^p(\partial\Omega)$. Finally, if $u \in C(\bar{\Omega}) \cap W^{1,p}(\Omega)$ then the $W^{1,p}$–approximating sequence of $C_0^\infty(\mathbb{R}^n)$ functions is also locally uniformly convergent (w.l.o.g. due to Theorem C.3). Thus, $u_n(x) \to u(x)$ for all $x \in \partial\Omega$. We know already that $Bu_n \to Bu$ in an $\mathbb{L}^q$ space on $\partial\Omega$ so that in particular a subsequence of Bu_n converges a.e. on $\partial\Omega$ to Bu which implies $Bu = u|_{\partial\Omega}$ and the proof is complete. ∎

Using the trace of $W^{1,p}$ functions, we can immediately define traces of derivatives. For $u \in W^{2,p}(\Omega)$, for example, the first derivatives are in $W^{1,p}(\Omega)$ and thus $B\frac{\partial u}{\partial x_i}$ is well defined. In particular, if n is the normal vector field on $\partial\Omega$ then the normal derivative $\partial u/\partial n$ is well defined for $u \in W^{2,p}(\Omega)$.

Another immediate consequence of Theorem C.4 is that the integration by parts formula can be extended to $W^{1,p}$ functions. If $\phi \in C_0^\infty(\mathbb{R}^n)$ and $u \in W^{1,p}(\Omega)$ with $1 \le p < \infty$ then

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx = \int_{\partial\Omega} (Bu)\phi n_i \, d\sigma - \int_\Omega \phi \frac{\partial u}{\partial x_i}\, dx. \tag{C.4}$$

(Note that for $C^{0,1}$ domains the outer unit normal field n is defined almost everywhere on $\partial\Omega$.) To prove (C.4) we approximate u by $C_0^\infty(\mathbb{R}^n)$ functions u_k. Then (C.4) holds with u_k in place of u due to Gauß theorem. Going over to the $W^{1,p}$ limit and noting that Bu_k converges in some $\mathbb{L}^q(\partial\Omega)$ to Bu so that $(Bu_k)\phi \to (Bu)\phi$ in $\mathbb{L}^1(\partial\Omega)$ we obtain (C.4). If we now take an approximating sequence $(\phi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ which approximates $v \in W^{1,q}(\Omega)$, we can pass to the limit in (C.4) with ϕ_k in place of ϕ which yields

$$\int_\Omega u \frac{\partial v}{\partial x_i}\, dx = -\int_\Omega v \frac{\partial u}{\partial x_i}\, dx + \int_{\partial\Omega} (Bu)(Bv) n_i \, d\sigma$$

for $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$. Similarly, one can show Green's formula for $u \in W^{1,p}(\Omega)$ and $v \in W^{2,q}(\Omega)$

$$\int_\Omega u \Delta v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} (Bu) n \cdot B(\nabla v)\, d\sigma.$$

In order to treat Dirichlet boundary problems, the closed subspace of functions in $W^{1,p}(\Omega)$ which have zero trace on $\partial\Omega$ (the kernel $\ker B$ of B) is especially important. Obviously, $C_0^\infty(\Omega) \subset \ker B$ and one can even show that the compactly supported C^∞ functions on Ω are dense in $\ker B$.

Lemma C.5. *Let Ω be a bounded $C^{0,1}$–domain in $\mathbb{R}^n$. Then, the closure of $C_0^\infty(\Omega)$ in the $W^{1,p}(\Omega)$ norm contains exactly those $W^{1,p}(\Omega)$ functions which vanish on $\partial\Omega$.*

Proof: Again, we refer to [**12**] for a detailed proof of this result and present the main idea only for the half space. We have already mentioned that the closure of $C_0^\infty(\Omega)$ certainly gives rise to $W^{1,p}(\Omega)$ functions which vanish on the boundary. Conversely, let $u \in W^{1,p}(\Omega)$ with $Bu = 0$. We define

$$v(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \Omega^c \end{cases}$$

Then, $v \in \mathbb{L}^p(\mathbb{R}^n)$ and the distributional x_i–derivative is given by the zero extension of $\partial u/\partial x_i$. Indeed, for $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\left\langle \frac{\partial v}{\partial x_i}, \phi \right\rangle = -\left\langle v, \frac{\partial \phi}{\partial x_i} \right\rangle = -\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx.$$

Using (C.4) we get

$$\left\langle \frac{\partial v}{\partial x_i}, \phi \right\rangle = -\int_{\partial\Omega} (Bu)\frac{\partial \phi}{\partial x_i} n_i \, d\sigma + \int_{\Omega} \frac{\partial u}{\partial x_i}\phi \, dx$$

and since $Bu = 0$ we get

$$\frac{\partial v}{\partial x_i}(x) = \begin{cases} \frac{\partial u}{\partial x_i} & x \in \Omega \\ 0 & x \in \Omega^c \end{cases}$$

In particular, the zero extension of a traceless $W^{1,p}(\Omega)$ function is contained in $W^{1,p}(\mathbb{R}^n)$. Since the translation operator

$$T_h v(x) = v(x + he_n)$$

is continuous on $\mathbb{L}^p(\mathbb{R}^n)$ (see [**1**]) it is also continuous on $W^{1,p}(\mathbb{R}^n)$ because

$$\frac{\partial}{\partial x_i}(T_h v) = T_h \frac{\partial v}{\partial x_i}.$$

The main idea of the proof is now to approximate v in $W^{1,p}(\mathbb{R}^n)$ by $T_{-h}v$ for a small $h > 0$. The shifted function $T_{-h}v$ is then supported inside the half space (a distance h away from the boundary) and can thus be approximated by $C_0^\infty(\Omega)$ functions (for example by cut–off and mollification). Consequently, v can be obtained as the $W^{1,p}(\Omega)$ limit of $C_0^\infty(\Omega)$ functions. ■

This characterization of $\ker B$ leads to a natural generalization. Even on domains Ω where a reasonable trace operator cannot be constructed, the closure of $C_0^\infty(\Omega)$ is well defined and contains, in some sense, functions which vanish on the boundary (because they can be approximated by functions which are zero on $\partial\Omega$). This gives rise to

Definition C.6. *Let $\Omega \subset \mathbb{R}^n$ be open and let $1 \le p \le \infty$, $m \in \mathbb{N}_0$. Then, $W_0^{m,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.*

Lemma C.5 now has the following counterpart.

Theorem C.7. *Let Ω be a bounded $C^{0,1}$–domain in $\mathbb{R}^n$, let $m \ge 1$ and $1 \le p < \infty$. Then,*

$$W_0^{m,p}(\Omega) = \{\, u \in W^{m,p}(\Omega) \,:\, B(\nabla^\alpha u) = 0, \quad \forall |\alpha| < m \,\}.$$

Proof: For $m = 1$ this is just Lemma C.5 so let us assume that the result is correct for $m-1$. If $u \in W_0^{m,p}$, then $\nabla^\beta u \in W_0^{1,p}(\Omega)$ if $|\beta| = m-1$. Thus $B(\nabla^\beta u) = 0$ according to Lemma C.5. By induction assumption and since $W_0^{m,p}(\Omega) \subset W_0^{m-1,p}(\Omega)$ we already have for all lower derivatives $B(\nabla^\alpha u) = 0$ for $|\alpha| < m-1$. Altogether,

$$u \in \{\, u \in W^{m,p}(\Omega) \,:\, B(\nabla^\alpha u) = 0, \quad \forall |\alpha| < m \,\}.$$

The converse direction is shown in a similar way based on Lemma C.5. ■

We remark that $Bu = 0$ implies automatically that all tangential derivatives of u vanish on the boundary (if the boundary is smooth enough to allow for tangential derivatives). Thus, the statement $Bu = 0$ and $B(\nabla u) = 0$, which holds for $W_0^{2,p}(\Omega)$ functions, is equivalent to $Bu = 0$ and $B(\partial u/\partial n) = 0$ since the derivatives in the remaining $n-1$ directions are tangential. Introducing the l^{th} normal derivative

$$\frac{\partial^l \phi}{\partial n^l}(x) := \left(\frac{d}{dt}\right)^l \phi(x + tn(x))\Bigg|_{t=0} = \sum_{|\alpha|=l} \frac{l!}{\alpha!} n^\alpha \nabla^\alpha \phi \qquad \phi \in C^\infty.$$

(observe that the t derivative is half sided with $t \le 0$) we thus have

Lemma C.8. *Let $1 \le p < \infty$, $m \in \mathbb{N}$ and let Ω be a bounded $C^{m,1}$ domain. Then $u \in W^{m,p}(\Omega)$ is contained in $W_0^{m,p}(\Omega)$ if and only if $\partial^l u/\partial n^l = 0$ a.e. on $\partial\Omega$ (in the sense of trace) for $l = 0, \ldots, m-1$.*

Proof: The idea is similar to the one of Lemma C.5 (see the reference cited in [**12**]). ■

C.3. Exercises

Exercise C.1. Show that $u(x) = \sin\frac{1}{x}$ is contained in $\mathbb{L}^p(0,1)$ but not in $W^{1,p}(0,1)$ for any $p \in [1,\infty]$. To which Sobolev space $W^{m,p}(0,1)$ does $v(x) = x^s$ belong for different values of $s \in \mathbb{R}$?

Exercise C.2. Let $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$ be the half space and let $u \in C_0^\infty(\mathbb{R}^n)$. Show (directly without using Sobolev embedding) that the $\mathbb{L}^2(\partial\Omega)$ norm of the boundary value of u can be estimated by the $W^{1,2}$ norm on Ω

$$\|Bu\|_{0,2,\partial\Omega} \le \|u\|_{1,2,\Omega}.$$

Exercise C.3. Let $\Omega \subset \mathbb{R}^n$ be open and $u \in W^{m,p}(\Omega)$ with $m \in \mathbb{N}$ and $p \in [1,\infty)$. Show that the zero extension of u

$$\tilde{u}(x) := \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \Omega^c \end{cases}$$

is an element of $W^{m,p}(\mathbb{R}^n)$ if u and its derivatives up to order $m-1$ vanish on the boundary (i.e. $u \in W_0^{m,p}(\Omega)$).

Exercise C.4.

a) Let $\phi_0(x) = 1$, $\phi_n(x) = \sqrt{2}\cos(n\pi x)$ for $n \in \mathbb{N}$. Show that $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\mathbb{L}^2(0,1)$.
(Hint: Check orthonormality; use the mapping $\cos(\pi\cdot) : [0,1] \mapsto [-1,1]$ and the Weierstraß approximation theorem to show that any $f \in C([0,1])$ can be uniformly approximated by expressions $\sum_{n=0}^N a_n(\cos(\pi x))^n$; conclude the proof with elementary trigonometric identities and the density of $C([0,1])$ in $\mathbb{L}^2(0,1)$.)

b) Let $\psi_n(x) = \sqrt{2}\sin(n\pi x)$ for $n \in \mathbb{N}$. Show that $\{\psi_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\mathbb{L}^2(0,1)$.
(Hint: After checking orthonormality, use that $f(x)/\sin(\pi x)$ is continuous on $[0,1]$ for any $f \in \mathscr{D}(0,1)$ together with the result in a), an elementary trigonometric identity, and the density of $\mathscr{D}(0,1)$ in $\mathbb{L}^2(0,1)$.)

c) Show that $\{e^{in\pi x}/\sqrt{2} : n \in \mathbb{Z}\}$ is a basis of $\mathbb{L}^2(-1,1)$.
(Hint: Use a) and b) and the decomposition of any function on $(-1,1)$ into an odd and even part.)

d) Use an appropriate normalization of the functions ψ_n in b) to construct an orthonormal basis of $H_0^1(0,1)$.

Exercise C.5. Let $s = \{(\gamma_n)_{n\in\mathbb{N}} : \gamma_n \in \mathbb{R}\}$ be the space of all real valued sequences. The subset of all terminating (or compactly supported) sequences is denoted

$$d = \{\gamma \in s : \gamma_n \neq 0 \quad \text{for finitely many } n\}.$$

A sequence $(\gamma^{(m)})_{m\in\mathbb{N}} \subset d$ is said to be convergent to γ in d if there exists $N \in \mathbb{N}$ such that $\gamma_n^{(m)} = 0$ for all $m \in \mathbb{N}$ and $n > N$ and $\gamma_n^{(m)} \to \gamma_n$ for $m \to \infty$ and $n \leq N$.

- Show that the dual d' can be identified with s based on the dual product

$$\langle \Lambda, \phi \rangle = \sum_{n\in\mathbb{N}} \Lambda_n \phi_n \qquad \forall \Lambda \in s, \quad \phi \in d.$$

(Note the similarity between $d, d' = s$ and $\mathscr{D}, \mathscr{D}'$.)

For $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $\gamma \in s$ we define

$$\|\gamma\|_{s,p} := \begin{cases} \left(\sum_{n\in\mathbb{N}} n^{sp} |\gamma_n|^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \max_{n\in\mathbb{N}} n^s |\gamma_n| & p = \infty \end{cases}$$

and set $w^{s,p} = \{\gamma \in s : \|\gamma\|_{s,p} < \infty\}$.

- Show that d is dense in $w^{s,p}$ for any $1 \leq p < \infty$ and $s \in \mathbb{R}$ but not if $p = \infty$.
- Show that $w^{s,p_1} \hookrightarrow w^{s,p_2}$ whenever $p_1 \leq p_2$ (i.e. $w^{s,p_1} \subset w^{s,p_2}$ and $\|\cdot\|_{s,p_2} \leq C\|\cdot\|_{s,p_1}$).
- Show that $w^{s_1,p_1} \hookrightarrow w^{s_2,p_2}$ is a compact embedding whenever $p_1 \leq p_2$ and $s_1 > s_2$.
- Show that $w^{s,\infty} \hookrightarrow w^{t,p}$ for $(s-t)p > 1$.