CHAPTER 1

Introduction

1.1. Classification of PDEs – some aspects

Generally speaking, partial differential equations (PDEs) characterize functions by relating their partial derivatives. To denote partial derivatives in a concise way, we introduce multi–index notation. A multi–index is a vector of nonnegative integers $\alpha \in \mathbb{N}_0^n$. If $x \in \mathbb{R}^n$ is some vector, the power x^α abbreviates

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

Applying this rule formally to the gradient operator

$$\nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)^T$$

we obtain the general differential operator (the variable x as subscript can be omitted in unambiguous cases)

$$\nabla_x^\alpha = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \dots \partial {x_n}^{\alpha_n}}$$

Here, we have set

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n.$$

By definition, the zero index gives rise to the identity operator $\nabla^{(0,\dots,0)} = \mathrm{id}$.

Using this notation we can now specify PDEs in the following way: Let $\Omega \subset \mathbb{R}^n$ be nonempty and open and let $\alpha^{(1)}, \dots, \alpha^{(M)}$ be a list of multi–indices. Given a function $F : \Omega \times \mathbb{R}^M \mapsto \mathbb{R}$, the expression

$$F(x, \nabla^{\alpha^{(1)}} u(x), \dots, \nabla^{\alpha^{(M)}} u(x)) = 0, \qquad x \in \Omega \tag{1.1}$$

is called *partial differential equation* for the function $u : \Omega \mapsto \mathbb{R}$. (Observe: the expression is an *ordinary differential equation* if x is a scalar variable, i.e. $n = 1$).

If u is vector valued, say in $\mathbb{R}^d$, the domain of F has to be changed to $\Omega \times (\mathbb{R}^d)^M$. Obviously, a system of PDEs is obtained if F is vector valued.

Our first classification of problem (1.1) is based on the linearity of the function F. If $F(x, v_1, \dots, v_M)$ is nonlinear in the v–variables, (1.1) is called a nonlinear PDE. If F is linear in v, we obtain a linear PDE which can be written in the form

$$L(x,\nabla)u = \sum_{|\alpha|\le m} a_\alpha(x)\nabla^\alpha u = 0. \tag{1.2}$$

Note that the coefficients $a_\alpha(x)$ need not be linear. By introducing zero coefficients, it is always possible to write the equation as sum over all multi-indices up to the highest appearing order.

Inbetween the fully linear and the nonlinear case one often introduces quasi–linear cases. They are characterized by F being linear at least in the variables corresponding to the highest u–derivatives.

Apart from the linearity of F, the choice of the multi-indices $\alpha^{(i)}$ also characterizes equation (1.1). In particular, the *order* of the equation

$$m = \max\{\, |\alpha^{(i)}| \,:\, i = 1,\ldots,M \,\}$$

generally influences the number of boundary conditions that have to be provided to make (1.1) uniquely solvable.

It turns out, however, that this crude classification with respect to linearity and order is not sufficient to describe the qualitative behavior of solutions to PDEs. Already within the class of second order, linear equations with constant coefficients, solutions differ considerably for example in their smoothness properties. A more refined classification scheme should identify classes of equations with common properties such that the type of an equation determines the nature of

- boundary and initial conditions which may be imposed,
- singularities which solutions may have,
- methods which can be used to approximate a solution.

In the following we briefly introduce the criterion which singles out *elliptic* equations.

Let us start with a second–order PDE in n space dimensions,

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0.$$

It turns out, that the operator L is effectively characterized by algebraic properties of the matrix $A(x) = (a_{ij}(x))$ (we can assume w.l.o.g. that $A(x)$ is symmetric). In particular, L (respectively $Lu = 0$) is called elliptic in $x \in \Omega$ if all eigenvalues of $A(x)$ have the same sign. In other words, ellipticity of L is related to definiteness of the matrix $A(x)$. (Observe that the curve $\{\, \xi \in \mathbb{R}^2 \,:\, \xi^T A \xi = 1 \,\}$ describes an *ellipse* if $A \in \mathbb{R}^{2\times 2}$ is positive definite.)

To generalize this characterization to equations of higher order we take a slightly different point of view. Given the general operator (1.2) of order m we define its *principal part* as the sum of the m^{th} order derivatives

$$L^p(x,\nabla) = \sum_{|\alpha|=m} a_\alpha(x)\nabla^\alpha.$$

The *symbol* of the principal part is then obtained by replacing ∇ with $i\xi$ where $\xi \in \mathbb{R}^n$ is a variable

$$L^p(x, i\xi) = \sum_{|\alpha|=m} i^m a_\alpha(x)\xi^\alpha, \qquad \xi \in \mathbb{R}^n.$$

For the second order operator we get in particular

$$L^p(x, i\xi) = -\xi^T A(x)\xi$$

which is just the quadratic form determined by the symmetric matrix $A(x)$. In accordance with the previous definition, we call the general operator $L(x, \nabla)$ elliptic in x iff

$$L^p(x, i\xi) = 0 \quad \Leftrightarrow \quad \xi = 0.$$

Finally, L is called elliptic if it is elliptic at each point $x \in \Omega$.

The definition of ellipticity for systems of equations is slightly more complicated and we refer to [**14**] for more details. The classification of nonlinear equations is typically reduced to the linear case by going over to the linearized equation.

In the remaining chapters we will see the importance of the ellipticity condition for both theoretical and numerical investigations. It gives rise to typical methods and thus really characterizes a coherent group of partial differential equations.

1.2. Examples of elliptic PDEs

The Laplacian

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is a paradigmatic elliptic operator (ellipticity is easily checked since the symbol of the principal part is just $\xi \mapsto -|\xi|^2$, which obviously satisfies the required condition). Using a suitable transformation of variables, the principal part of any second order linear elliptic operator with constant coefficients turns into a Laplacian. In this sense, the Laplacian is a *normal form* for these operators and its investigation yields results which apply to the whole family.

In the following section, we present some exemplary physical problems where the Laplacian plays an important role. After that, we list some classical properties of harmonic functions, i.e. solutions of the equation $\Delta u = 0$.

1.2.1. Examples arising in physics. If we consider Maxwell's equations for the electric field E and the magnetic field B in vacuum ($x \in \mathbb{R}^3$)

$$\begin{aligned} \operatorname{curl} E &= -\frac{\partial}{\partial t}B, & \operatorname{div} E &= \rho, \\ \operatorname{curl} B &= \frac{\partial}{\partial t}E + j, & \operatorname{div} B &= 0, \end{aligned}$$

we find in the steady case (i.e vanishing time derivatives)

$$\operatorname{curl} E = 0, \quad \operatorname{div} E = \rho, \qquad x \in \mathbb{R}^3.$$

Here, the charge density ρ is the source of the electric field. Since $\operatorname{curl} E = 0$, we can introduce an electric potential Φ such that $E = \nabla\Phi$. Inserting this relation in the divergence condition yields the so called *Poisson equation*

$$\Delta\Phi = \rho.$$

In the special case $\rho = 0$, we obtain Laplace's equation

$$\Delta\Phi = 0$$

for the electric potential. We stress that both equations are obtained in a steady situation. In fact, elliptic equations typically appear as time independent problems (although time independence does not imply ellipticity in general).

Another source for elliptic equations are conservation laws, i.e equations of the form

$$\frac{\partial u}{\partial t} + \operatorname{div} q(u) = f.$$

In steady state, the equation turns into

$$\operatorname{div} q(u) = f$$

and it is elliptic if the flux $q(u)$ is proportional to the gradient of u, i.e. $q(u) = \lambda\nabla u$. If $\lambda \neq 0$ is a constant, we get again Poisson equation

$$\lambda\Delta u = f.$$

As example, we mention heat transfer problems. Here, u is the heat density which can be expressed in terms of temperature T, specific heat c and density ρ of the material

$$u = c\rho T.$$

According to Fourier's law, the heat flux vector q points in the opposite direction of the temperature gradient

$$q = -k\nabla T$$

where k denotes the heat conductivity. If there are no heat sources ($f = 0$), the conservation law for u yields in the steady case

$$\operatorname{div}(k\nabla T) = 0.$$

If k is a constant, this is just Laplace's equation. If k is a differentiable function of position, we get by product rule

$$\operatorname{div}(k\nabla T) = k\Delta T + \nabla T \cdot \nabla k.$$

Again, the principal part of this operator is $k\Delta$, which is elliptic if $k \neq 0$ everywhere.

A similar situation occurs in diffusion processes. If c is the concentration of some material contained in a medium, the diffusion of c is described by Fick's law which relates the mass flux q of the concentration c by the relation

$$q = -D\nabla c.$$

Due to conservation of mass, c obeys a conservation law, so that in steady state

$$\operatorname{div}(D\nabla c) = 0.$$

Our everyday experience with heat conduction or diffusion processes (switching on a heater, putting milk in coffee, etc.) already gives us some insight into the qualitative behavior of solutions to Laplace's equation. In particular, a room which has been heated for a long time (steady state) does not show jumps in temperature – instead, the temperature distribution is smooth. The same is true in diffusion processes where jumps in concentration are smeared out so that in steady state the concentration varies smoothly.

1.2.2. Some properties of harmonic functions. In this section we present properties of solutions to Laplace's equation $\Delta u = 0$.

Definition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be open. Any function $u \in C^2(\Omega)$ which satisfies $\Delta u = 0$ in Ω is called harmonic.*

Simple examples of harmonic functions are affine linear maps. A more instructive example is obtained with the constraint that the harmonic function should be radially symmetric, i.e. $u(x) = \bar{g}(|x|)$. Using the derivatives of $r(x) = |x|$

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \qquad \frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{1}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)$$

this implies

$$\frac{d^2\bar{g}}{dr^2} + \frac{n-1}{r}\frac{d\bar{g}}{dr} = 0.$$

The general solution of this equation is given by

$$\bar{g}(r) = c_1 + c_2 g(r), \tag{1.3}$$

where

$$g(r) = \begin{cases} \frac{1}{\omega_1} r & n = 1 \\ \frac{1}{\omega_2} \ln r & n = 2 \\ \frac{1}{(2-n)\omega_n} r^{2-n} & n > 2. \end{cases} \tag{1.4}$$

Although g has a different structure in different dimensions, the derivative is always of the form

$$g'(r) = \frac{1}{\omega_n}\frac{1}{r^{n-1}}.$$

The scaling constant ω_n is chosen in view of later applications. It represent the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$ which is in terms of the gamma function

$$\omega_n = \frac{2\Gamma\left(\frac{1}{2}\right)^n}{\Gamma\left(\frac{n}{2}\right)}, \quad \omega_1 = 2, \quad \omega_2 = 2\pi, \quad \omega_3 = 4\pi.$$

Note that in the one dimensional case the unit ball is just the interval $(-1, 1)$ so that the unit sphere S^0 consists of the two points $x = \pm 1$. If we use the surface measure

$$\sigma = \delta_{-1} + \delta_{+1},$$

surface integrals reduce to sums

$$\int_{S^0} f(x)\, d\sigma = f(-1) + f(+1).$$

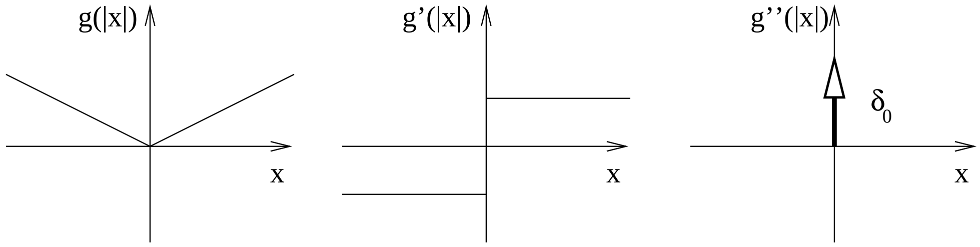
In particular, the surface volume ω_1 is given by

$$\omega_1 = \int_{S^0} d\sigma = 2.$$

We also remark that Gauss' theorem reduces to the fundamental theorem of calculus if we adopt the above convention for the surface measure. More precisely, let $\Omega = (a, b)$ be some interval with boundary $\partial\Omega = \{a, b\}$ and surface measure $\sigma = \delta_a + \delta_b$. Since the divergence of a function $f \in C^1(\bar{\Omega})$ is just the derivative $f'(x)$ and the outward normal field on $\partial\Omega$ is given by $n(a) = -1$ and $n(b) = +1$, we find

$$\int_a^b f'(x)\, dx = \int_\Omega \operatorname{div} f(x)\, dx = \int_{\partial\Omega} f(x) n(x)\, d\sigma = f(b) - f(a).$$

Going back to (1.4), we stress that $g(|x|)$ is not harmonic in $\mathbb{R}^n$ due to the singularity in $x = 0$. However, away from the origin, g is C^∞ smooth and satisfies $\Delta g(|x|) = 0$. The behavior of g at the singularity is most easily seen in the one dimensional case.



We find the important relation

$$\Delta g(|x|) = \delta_0 \tag{1.5}$$

which will be proved rigorously in Lemma 1.4 with the help of Green's formula. These formulas play an important role in the investigation of harmonic functions. They will also be frequently used in later chapters.

Lemma 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a piecewise C^1–boundary whose unit outward normal field is n. For $u, v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$, Green's formula holds*

$$\int_\Omega u\Delta v - v\Delta u \, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \, d\sigma.$$

Proof: Using the identity

$$\operatorname{div}(u\nabla v) = u\Delta v + \nabla u \cdot \nabla v$$

and Gauss' theorem, we get immediately

$$\int_\Omega u\Delta v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} u\frac{\partial v}{\partial n} \, d\sigma.$$

Changing the role of u and v and subtracting the formulas leads to Green's formula since the symmetric term $\int_\Omega \nabla u \cdot \nabla v \, dx$ disappears. ■

As immediate consequence we obtain

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a piecewise C^1–boundary whose unit outward normal field is n. If $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ is harmonic then*

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \, d\sigma = 0.$$

Proof: Just set $v \equiv 1$ in Green's formula. ■

With the help of Green's formula, we can prove an auxiliary lemma which allows us to deduce another property of harmonic functions. The lemma is a reformulation of (1.5) which does not refer explicitly to the Dirac delta distribution.

Lemma 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a piecewise C^1–boundary. If $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ then for any $x \in \Omega$*

$$u(x) = \int_{\partial\Omega} u(y)\frac{\partial}{\partial n_y}G(x,y) - G(x,y)\frac{\partial}{\partial n}u(y) \, d\sigma + \int_\Omega G(x,y)\Delta u(y) \, dy$$

where

$$G(x,y) = g(|x-y|) = \begin{cases} \frac{1}{\omega_1}|x-y| & n = 1 \\ \frac{1}{\omega_2}\ln|x-y| & n = 2 \\ \frac{1}{(2-n)\omega_n}|x-y|^{2-n} & n > 2. \end{cases}$$

Proof: The basic idea of the proof is as follows (note that this is just a formal argument): Granting that $x \mapsto G(x,0) = g(|x|)$ solves

$$\Delta G(x,0) = \delta_0(x),$$

we get

$$\Delta_y G(x,y) = \Delta_y g(|x-y|) = \delta_0(y-x)$$

so that

$$u(x) = \int_\Omega u(y)\text{"}\Delta_y G(x,y)\text{"}\, dy.$$

Applying Green's formula (only formally (!) since G is not in $C^2(\Omega)$) we get

$$u(x) = \int_{\partial\Omega} u(y)\frac{\partial}{\partial n_y}G(x,y) - G(x,y)\frac{\partial}{\partial n}u(y)\, d\sigma + \int_\Omega G(x,y)\Delta u(y)\, dy$$

which is the desired equality. A rigorous proof of the statement essentially avoids direct integration of the singularity of $\Delta_y G(x,y)$ at $y = x$. This is achieved by cutting out a ball B_ϵ of radius $\epsilon > 0$ around $x \in \Omega$. Since $y \mapsto G(x,y)$ is obviously contained in $C^1(\bar\Omega_\epsilon) \cap C^2(\Omega_\epsilon)$ with $\Omega_\epsilon = \Omega \backslash \bar B_\epsilon$, we can apply Green's formula in the modified domain Ω_ϵ

$$\begin{aligned}\int_{\Omega_\epsilon} & u(y)\Delta_y G(x,y) - G(x,y)\Delta u(y)\, dy \\ &= \int_{\partial\Omega_\epsilon} u(y)\frac{\partial}{\partial n_y}G(x,y) - G(x,y)\frac{\partial}{\partial n}u(y)\, d\sigma.\end{aligned}$$

Splitting $\partial\Omega_\epsilon$ into $\partial\Omega$ and ∂B_ϵ and using $\Delta_y G(x,y) = 0$ in Ω_ϵ we get

$$\begin{aligned}\int_{\partial B_\epsilon} & u(y)\frac{\partial}{\partial n_y}G(x,y) - G(x,y)\frac{\partial}{\partial n}u(y)\, d\sigma \\ &= \int_{\partial\Omega} u(y)\frac{\partial}{\partial n_y}G(x,y) - G(x,y)\frac{\partial}{\partial n}u(y)\, d\sigma + \int_{\Omega_\epsilon} G(x,y)\Delta u(y)\, dy.\end{aligned}$$

Observe that the outer normal of $\partial\Omega_\epsilon$ on the boundary ∂B_ϵ points into the ball B_ϵ. By changing the sign of the integral, however, we have used $\frac{\partial}{\partial n}$ to denote the derivative with respect to the outer normal of ∂B_ϵ. Using the definition of $G(x,y)$, we can easily calculate the normal derivative

$$\begin{aligned}\frac{\partial}{\partial n_y}G(x,y) &= n(y)\cdot\nabla_y G(x,y) \\ &= \frac{y-x}{|y-x|}\cdot\left(g'(|x-y|)\frac{y-x}{|y-x|}\right) = \frac{1}{\omega_n}\frac{1}{|x-y|^{n-1}}.\end{aligned}$$

Due to continuity of u we can thus infer

$$\begin{aligned}\int_{\partial B_\epsilon} & u(y)\frac{\partial}{\partial n_y}G(x,y)\, d\sigma \\ &= \frac{1}{\omega_n}\frac{1}{\epsilon^{n-1}}\int_{\partial B_\epsilon} u(y)\, d\sigma = \frac{1}{|\partial B_\epsilon|}\int_{\partial B_\epsilon} u(y)\, d\sigma \underset{\epsilon\to 0}{\longrightarrow} u(x).\end{aligned}$$

Since $y \mapsto G(x,y)$ is locally integrable in $\mathbb{R}^n$ (the singularity in x is *weak*), we also get

$$\int_{\Omega_\epsilon} G(x,y)\Delta u(y)\, dy \underset{\epsilon\to 0}{\longrightarrow} \int_\Omega G(x,y)\Delta u(y)\, dy$$

(by Lebesgue's dominated convergence theorem). Since the singularity of G is even integrable on the sphere in $\mathbb{R}^n$, the remaining integral also vanishes. Transformation to the unit sphere S^{n-1} yields

$$\int_{\partial B_\epsilon} G(x,y)\frac{\partial}{\partial n}u(y)\,d\sigma = g(\epsilon)\epsilon^{n-1}\frac{1}{\omega_n}\int_{S^{n-1}}\frac{\partial}{\partial n_\xi}u(x+\epsilon\xi)\,d\sigma.$$

While the integral is bounded, the leading factor tends to zero as $\epsilon \to 0$. ■

An important conclusion which we draw from Lemma 1.4 is

Theorem 1.5. (Mean Value Property)
Let u be harmonic in $\Omega \subset \mathbb{R}^N$ and let $\overline{B_R(x_0)} \subset \Omega$ be a ball around some point $x_0 \in \Omega$. Then $u(x_0)$ is equal to its spherical mean

$$u(x_0) = \frac{1}{|\partial B_R(x_0)|}\int_{\partial B_R(x_0)} u\,d\sigma = \frac{1}{\omega_n}\int_{S^{n-1}} u(x_0+R\xi)\,d\sigma(\xi)$$

and also to its mean value in $B_R(x_0)$

$$u(x_0) = \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u\,dx = \frac{n}{\omega_n}\int_{B_1(0)} u(x_0+Ry)\,dy.$$

Proof: Since u is harmonic, we obtain from Lemma 1.4

$$u(x_0) = \int_{\partial B_R(x_0)} u(y)\frac{\partial}{\partial n_y}g(|x_0-y|) - g(|x_0-y|)\frac{\partial}{\partial n}u(y)\,d\sigma(y).$$

On the surface of the ball $B_R(x_0)$ we have $|x_0 - y| = R$ so that with Corollary 1.3

$$\int_{\partial B_R(x_0)} g(R)\frac{\partial}{\partial n}u(y)\,d\sigma(y) = 0.$$

The normal derivative $\frac{\partial}{\partial n_y}g(|x_0-y|)$ is just

$$\frac{\partial}{\partial n_y}g(|x_0-y|) = g'(R)\frac{y-x_0}{|y-x_0|}\cdot\frac{y-x_0}{|y-x_0|} = g'(R) = \frac{1}{\omega_n R^{n-1}}.$$

Since $\omega_n R^{n-1}$ is the surface volume of $\partial B_R(x_0)$ we obtain

$$u(x_0) = \frac{1}{|\partial B_R(x_0)|}\int_{\partial B_R(x_0)} u\,d\sigma.$$

Finally, a transformation of variables yields

$$u(x_0) = \frac{1}{\omega_n}\int_{S^{n-1}} u(x_0+R\xi)\,d\sigma(\xi).$$

Since there has been no restriction on the radius R other than the ball being contained in Ω, we also get for any $0 < \rho < 1$

$$u(x_0) = \frac{1}{\omega_n}\int_{S^{n-1}} u(x_0+\rho R\xi)\,d\sigma(\xi). \tag{1.6}$$

Multiplying (1.6) by ρ^{n-1} and integrating over $(0,1)$ leads to

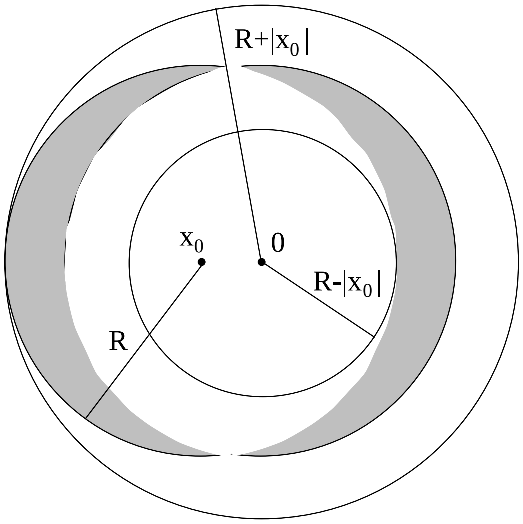
$$\frac{1}{n}u(x_0) = \frac{1}{\omega_n}\int_{B_1(0)} u(x_0 + Ry)\, dy.$$

■

Applying the mean value theorem to $\Omega = \mathbb{R}^n$ we obtain the interesting result that the only bounded harmonic functions on the whole space are the constant functions. To see this, we compare the values of a harmonic function at two points, say at $x = x_0$ and the origin $x = 0$. Choosing a ball of very large radius R around these two points, we have

$$u(0) = \frac{1}{|B_R(0)|}\int_{B_R(0)} u(x)\, dx \qquad \text{and} \qquad u(x_0) = \frac{1}{|B_R(0)|}\int_{B_R(x_0)} u(x)\, dx.$$

If $R > |x_0|$, the difference Σ in the domains of integration has a shape as depicted in the figure below (the shaded part).



Clearly, the volume on which the domains differ is confined inbetween the balls $B_{R-|x_0|}(0)$ and $B_{R+|x_0|}(0)$. It can be estimated by

$$|\Sigma| \le |B_{R+|x_0|}(0)\backslash B_{R-|x_0|}(0)| = \frac{\omega_n}{n}\left(|R+|x_0||^n - |R-|x_0||^n\right) = \mathcal{O}(R^{n-1}).$$

Taking into account the scaling by $|B_R(0)| \sim R^n$ and the boundedness of u, we get

$$|u(x_0) - u(0)| \le \frac{1}{|B_R(0)|}\int_\Sigma |u(x)|\, dx = \mathcal{O}\left(\frac{1}{R}\right).$$

In the limit $R \to \infty$ we find $u(x_0) = u(0)$ for any $x_0 \in \mathbb{R}^n$. The result is summarized in the following theorem.

Theorem 1.6. *Let u be harmonic on $\mathbb{R}^n$ and bounded. Then u is constant.*

The next conclusion which we draw from the mean value theorem is

Theorem 1.7. (Maximum Principle)
Let u be harmonic on the open, connected set $\Omega \subset \mathbb{R}^n$ with $\sup_{x\in\Omega} u(x) = A < \infty$. Then either $u(x) < A$ for all $x \in \Omega$ or $u(x) = A$ on Ω.

Proof: The set $\Lambda := \{x \in \Omega : u(x) = A\}$ is relatively closed in Ω since u is continuous. Our aim is to show that Λ is at the same time open. Since Ω is assumed to be connected, this implies that $\Lambda = \Omega$ or $\Lambda = \emptyset$ and thus yields the result. To show that Λ is open, we pick $x_0 \in \Lambda$. For $R > 0$ sufficiently small, we get $B_R(x_0) \subset \Omega$ and by the mean value theorem

$$A = u(x_0) = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(y)\, dy. \tag{1.7}$$

If, somewhere in the ball $B_R(x_0)$, $u(y)$ is less than the maximum A of u, then also the average is less than A so that (1.7) implies $B_R(x_0) \subset \Lambda$ which completes the proof. ■

In particular, if Ω is bounded and u is continuous on $\bar{\Omega}$, then u attains its maximum somewhere on the compact set $\bar{\Omega}$. According to the maximum principle, u is either constant or does not attain its maximum in Ω. Altogether, the maximum is attained on the boundary of Ω.

Corollary 1.8. (weak maximum principle) *If Ω is bounded and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω then the maximum and the minimum of u is attained on the boundary $\partial\Omega$.*

Proof: If u is harmonic, also $-u$ is harmonic and we find with the above argument that $\max_{x\in\bar{\Omega}}(-u(x))$ is attained at some point $x_0 \in \partial\Omega$. Since $-\max(-u) = \min u$, the function u is minimal in x_0 which shows the result concerning the minimum. We note that also Theorem 1.7 can be formulated for the minimum by going over from u to $-u$. ■

In Chapter 2 we will investigate maximum principles for more general elliptic operators. They turn out to be useful to show uniqueness of solutions as well as continuous dependence of the solution on the boundary data.

To complete this section, we will now prove an inverse version of the mean value theorem.

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C(\Omega)$. If for all $x_0 \in \Omega$ and $R > 0$ such that $\overline{B_R(x_0)} \subset \Omega$, the spherical mean value property*

$$u(x_0) = \frac{1}{|\partial B_R(x_0)|} \int_{\partial B_R(x_0)} u(x)\, d\sigma(x)$$

is satisfied, then u is harmonic and contained in $C^\infty(\Omega)$.

Proof: We pick a rotationally symmetric function $\phi(x) = \psi(|x|) \geq 0$ which is supported on the unit ball $B_1(0) \subset \mathbb{R}^n$ and is C^∞ smooth (a so called *mollifier*). The integral over ϕ is supposed to be normalized to one. After rescaling

$$\phi_\epsilon(x) := \frac{1}{\epsilon^n}\phi\left(\frac{x}{\epsilon}\right)$$

the latter condition remains true but the support shrinks to the ball $B_\epsilon(0)$. For some given $\epsilon > 0$ we now define the set Ω_ϵ as those points in Ω whose distance to the boundary is at least ϵ. On Ω_ϵ we can then convolute the given function u with ϕ_ϵ and obtain

$$\tilde{u}(x) := \int_{\mathbb{R}^n} u(y)\phi_\epsilon(x-y)\,dy, \qquad x \in \Omega_\epsilon.$$

Since $\phi_\epsilon \in C^\infty$, also $\tilde{u} \in C^\infty(\Omega_\epsilon)$. We will show that $u = \tilde{u}$ on Ω_ϵ if u satisfies the mean value property. Since ϵ can be chosen arbitrarily small, this implies $u \in C^\infty(\Omega)$. Taking $x \in \Omega_\epsilon$, we get

$$\begin{aligned}\tilde{u}(x) &= \int_{\mathbb{R}^n} u(x-y)\phi_\epsilon(y)\,dy = \int_{B_1(0)} u(x-\epsilon z)\psi\left(|z|\right)\,dz \\ &= \int_0^1 \left(\int_{S^{n-1}} u(x-\epsilon r\xi)\,d\sigma(\xi)\right)\psi(r)r^{n-1}\,dr \\ &= \omega_n u(x)\int_0^1 \psi(r)r^{n-1}\,dr.\end{aligned}$$

In the last step, the mean value property has been used. Doing all transformations backwards without u in the integral, we get

$$\tilde{u}(x) = u(x)\int_{\mathbb{R}^n} \phi_\epsilon(x)\,dx = u(x).$$

To show that u is also harmonic, we apply Green's formula with $v \equiv 1$ on the domain $B_R(x_0)$

$$\int_{B_R(x_0)} \Delta u\,dy = \int_{\partial B_R(x_0)} \frac{\partial u}{\partial n}\,d\sigma = \int_{S^{n-1}} \frac{\partial u}{\partial n}(x_0+R\xi)\,d\sigma(\xi).$$

Since

$$\left.\frac{d}{dr}u(x_0+r\xi)\right|_{r=R} = \nabla_x u(x_0+R\xi)\cdot\xi = \frac{\partial u}{\partial n}(x_0+R\xi)$$

we get again with the mean value property

$$\int_{B_R(x_0)} \Delta u\,dy = \left.\frac{d}{dr}\int_{S^{n-1}} u(x_0+r\xi)\,d\sigma(\xi)\right|_{r=R} = \frac{d}{dr}(\omega_n u(x_0)) = 0.$$

After dividing by the volume of $B_R(x_0)$, we obtain in the limit $R \to 0$ that $\Delta u(x_0) = 0$ and hence $\Delta u = 0$ in Ω. ∎

Instead of requiring the spherical mean value property, the same result follows if we assume

$$u(x_0) = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(y)\, dy.$$

Indeed, this implies the spherical mean value property (Exercise 1.8). Finally, we note that any harmonic function is automatically C^∞ smooth. Since harmonic functions have the mean value property (Theorem 1.5), the smoothness follows from Theorem 1.9. Also, the limit of a sequence of harmonic functions which converges uniformly on $\bar{\Omega}$ is again harmonic.

Corollary 1.10. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of harmonic functions on Ω which converges locally uniformly on Ω. Then, the limit function is again harmonic.*

Proof: Locally, the limit u satisfies for any $x \in \Omega$ and $R > 0$ such that $\overline{B_R(x_0)} \subset \Omega$.

$$\begin{aligned} u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} u_n(y)\, d\sigma(y) \\ = \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} u(y)\, d\sigma(y) \end{aligned}$$

Hence, u has the mean value property and thus is harmonic according to Theorem 1.9. ■

One can even show that harmonic functions are analytic [**10**].

1.3. Elementary solution concepts

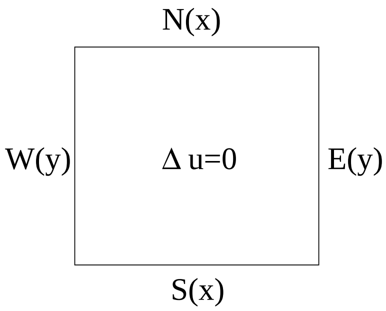
1.3.1. Separation of variables. The idea of this method is to reduce a PDE to a system of ODEs and expressing the solution of the PDE as a sum or infinite series. To illustrate the method, we consider Laplace's equation on the unit square

$$\Omega = \{\, (x, y) \in \mathbb{R}^2 \;:\; 0 < x < 1,\, 0 < y < 1 \,\}.$$

We wish to find a function $u : \bar{\Omega} \mapsto \mathbb{R}$ satisfying

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \tag{1.8}$$

at each point in Ω, having the values $N(x), E(y), S(x), W(y)$ at north, east, south and west face respectively. Moreover, the boundary values are assumed to be continuous including the corners, i.e. $W(1) = N(0)$, $N(1) = E(1)$, etc.



The key to separation of variables is to look for solutions of (1.8) of the form

$$u(x, y) = X(x)Y(y).$$

When we put a function of this form into (1.8), the partial derivatives in the differential equation appear as ordinary derivatives of the functions X and Y, i.e. (1.8) becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

At any point (x, y) at which u is nonzero, we can divide this equation by u and rearrange to get

$$\frac{X''(x)}{X(x)} = -\frac{Y''(x)}{Y(x)}.$$

Since the right hand side of the equation does not depend on x, neither can the left hand side. Using the same argument for y, we find that both quotients are independent of x and y, i.e. they are constant on Ω. Denoting this constant λ, we obtain

$$X'' = \lambda X,$$
$$Y'' = -\lambda Y.$$

Solving these equations, we get the following four parameter family of solutions of (1.8)

$$u(x, y) = A\left(e^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}\right)\left(e^{\sqrt{-\lambda}y} + Ce^{-\sqrt{-\lambda}y}\right).$$

The parameters now have to be chosen in such a way that the prescribed boundary conditions are met. Before we go into that, we simplify the problem by splitting it up into four sub problems of the same structure but with the assumption that only one of the functions N, E, S, W is not identically zero. If the corresponding solutions are denoted u_N, u_E, u_S, u_W, the overall solution is simply the sum

$$u = u_N + u_E + u_S + u_W$$

due to the linearity of the problem, To calculate u_N, we note that the zero boundary conditions on eastern, southern and western face are obtained by setting $\lambda = -(m\pi)^2$, $B = -1$, $C = -1$, i.e.

$$u_N(x, y) = A\sin(m\pi x)\sinh(m\pi y), \qquad m \in \mathbb{N}.$$

Now, if $N(x)$ is of the form

$$N(x) = \sum_{m=1}^{M} \alpha_m \sin(m\pi x)$$

we can simply take

$$A_m := \frac{\alpha_m}{\sinh(m\pi)}, \qquad m = 1, \ldots, M$$

and define

$$u_N(x,y) = \sum_{m=1}^{M} A_m \sin(m\pi x)\sinh(m\pi y).$$

From the theory of Fourier series, we know that any function in $\mathbb{L}^2(0,1)$ can be expressed in that way if we allow $M = \infty$. However, the series converges in general not point wise which is most easily seen at the endpoints $x = 0$ and $x = 1$ (e.g. $N(x) \equiv 1$ can not be approximated in these points). On the other hand, one can show that the convergence is uniform if N is sufficiently smooth and vanishes at the endpoints. Note that we can assume without loss of generality that $N(0) = N(1) = 0$. The trick is to subtract a function which satisfies Laplace's equation and satisfies the boundary conditions in the corners. As example we mention

$$\hat{u}(x,y) = S(0)(1-x)(1-y) + S(1)x(1-y) + N(0)(1-x)y + N(1)xy.$$

Altogether, we can write

$$u(x,y) = \hat{u}(x,y) + \sum_{m=1}^{\infty} \big\{ \hat{N}_m \sin(m\pi x)\sinh(m\pi y) + \hat{E}_m \sinh(m\pi x)\sin(m\pi y) \\ + \hat{S}_m \sin(m\pi x)\sinh(m\pi(1-y)) + \hat{W}_m \sinh(m\pi(1-x))\sin(m\pi y) \big\}$$

where

$$\hat{N}_m = \frac{2}{\sinh(m\pi)} \int_0^1 (N(x) - \hat{u}(x,1))\sin(m\pi x)\,dx,$$
$$\hat{E}_m = \frac{2}{\sinh(m\pi)} \int_0^1 (E(y) - \hat{u}(1,y))\sin(m\pi y)\,dy,$$
$$\hat{S}_m = \frac{2}{\sinh(m\pi)} \int_0^1 (S(x) - \hat{u}(x,0))\sin(m\pi x)\,dx,$$
$$\hat{W}_m = \frac{2}{\sinh(m\pi)} \int_0^1 (W(y) - \hat{u}(0,y))\sin(m\pi y)\,dy.$$

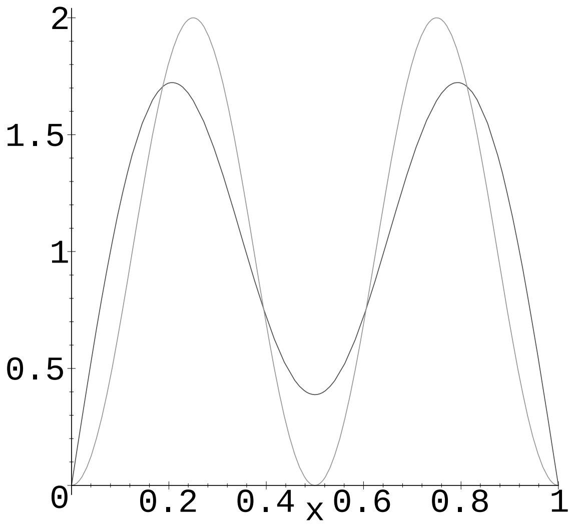
As example, we consider the case where N, E, S, W are restrictions of the function

$$b(x,y) = \cos(4\pi x)\cos(3\pi y)$$

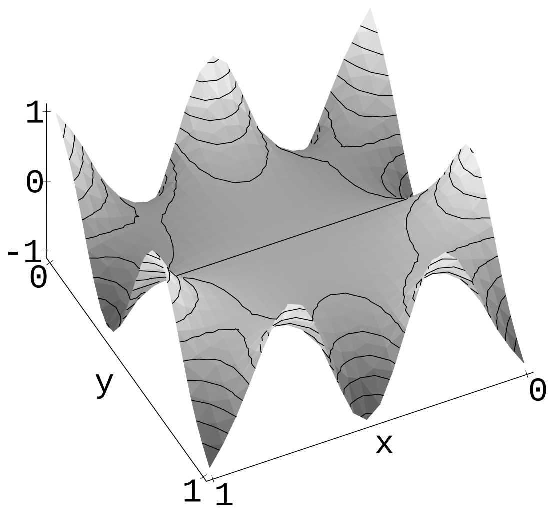
to the boundary. Truncating the Fourier series at $m = 3$, the approximation of the boundary values is still poor. After subtraction of

$$\hat{u}(x, y) = (1 - x)(1 - y) + x(1 - y) - (1 - x)y - xy$$

we get for example at the northern face



where the solid curve is the approximation. Truncating at $m = 5$ the graphs almost coincide and we get a good approximation of the solution of Laplace's equation.



1.3.2. Solution by integral operators. In order to find a solution of Poisson equation

$$\Delta\Phi(x) = \rho(x), \qquad x \in \mathbb{R}^n,\ n \geq 2, \tag{1.9}$$

we try the integral representation

$$\Phi(x) = \int_{\mathbb{R}^n} K(x,y)\rho(y)\,dy.$$

If such a function K exists (the so called *kernel* of the integral operator), it must be of the form

$$K(x,y) = k(|x-y|), \qquad x,y \in \mathbb{R}^n \tag{1.10}$$

with a suitable function k. This property follows from the fact that both the underlying space $\mathbb{R}^n$ and the Laplacian are invariant under translations and rotations. Translation invariance of Δ means nothing else but

$$\Delta_x\big(\Psi(x-z)\big) = (\Delta_x\Psi)(x-z), \qquad \forall x,z \in \mathbb{R}^n.$$

Now, if Φ solves (1.9) then $x \mapsto \Phi(x-z)$ solves (1.9) with respect to the shifted source $\rho(x-z)$. Consequently,

$$\Phi(x-z) = \int_{\mathbb{R}^n} K(x-z,y)\rho(y)\,dy = \int_{\mathbb{R}^n} K(x,y)\rho(y-z)\,dy,$$

so that

$$\int_{\mathbb{R}^n} \big(K(x-z,y) - K(x,y+z)\big)\rho(y)\,dy = 0.$$

Since ρ can be chosen arbitrarily from a sufficiently large class of functions (say $C_0^\infty(\mathbb{R}^n)$), we get

$$K(x,y+z) = K(x-z,y).$$

Setting $y=0$, we finally obtain

$$K(x,z) = K(x-z,0).$$

As far as rotational invariance is concerned, we note that for any rotation matrix $R \in SO(n)$ with Einstein's summation convention on repeated indices

$$\frac{\partial^2}{\partial x_i^2}\big(\Psi(Rx)\big) = \frac{\partial}{\partial x_i}\left(\frac{\partial \Psi}{\partial x_k}(Rx)R_{ki}\right) = \frac{\partial^2\Psi}{\partial x_k \partial x_l}(Rx)R_{ki}R_{li} = \frac{\partial^2\Psi}{\partial x_k^2}(Rx).$$

Again, if Φ solves (1.9) then $x \mapsto \Phi(Rx)$ is a solution with respect to the rotated source $\rho(Rx)$. With a similar argument as for the translation, we get

$$K(x, R^{\mathrm{T}}y) = K(Rx,y).$$

Setting $y=0$, we find for any $x \in \mathbb{R}^n$

$$K(x,0) = K(|x|e_1, 0)$$

by choosing a rotation matrix which maps x onto the first unit vector e_1. A combination of translational and rotational invariance thus leads to (1.10) by setting

$$k(r) = K(re_1, 0).$$

To find out more about the structure of K, we formally replace the regular function $\rho(x)$ by a Dirac delta distribution $\delta_0(x)$. Due to our ansatz, the resulting function

$$\Phi(x) = \int_{\mathbb{R}^n} k(|x-y|)\text{"}\delta_0(y)\text{"}\, dy = k(|x|)$$

then solves

$$\Delta k(|x|) = \delta_0(x).$$

(All the above can be made precise in the framework of generalized functions which we introduce in the appendix.) The solution of this problem has already been determined in Section 1.2.2 where it is denoted $g(|x|)$. Instead of validating each step in the above derivation, we just take it as a motivation and check directly that the constructed integral operator yields a solution of (1.9).

Theorem 1.11. (Existence)
Let $\rho \in C_0^2(\mathbb{R}^n)$. Then

$$\Phi(x) = \int_{\mathbb{R}^n} G(x,y)\rho(y)\, dy$$

is a solution of the Poisson problem

$$\Delta\Phi(x) = \rho(x), \qquad x \in \mathbb{R}^n. \tag{1.11}$$

The function G is defined by

$$G(x,y) = g(|x-y|) = \begin{cases} \frac{1}{\omega_1}|x-y| & n=1 \\ \frac{1}{\omega_2}\ln|x-y| & n=2 \\ \frac{1}{(2-n)\omega_n}|x-y|^{2-n} & n>2 \end{cases}$$

Proof: First let us remark that the integral is well defined since ρ is compactly supported and bounded. Rewriting the expression as

$$\Phi(x) = \int_{\mathbb{R}^n} g(|z|)\rho(x-z)\, dz$$

we can obviously differentiate under the integral

$$\begin{aligned}\Delta\Phi(x) &= \int_{\mathbb{R}^n} g(|z|)\Delta_x\rho(x-z)\, dz \\ &= \int_{\mathbb{R}^n} g(|z|)(\Delta_z\rho)(x-z)\, dz = \int_{\mathbb{R}^n} G(x,y)\Delta\rho(y)\, dy.\end{aligned}$$

Since ρ is supported inside a ball $B_R(0)$ of sufficiently large radius $R>0$, we get

$$\Delta\Phi(x) = \int_{B_R(0)} G(x,y)\Delta\rho(y)\, dy.$$

Applying Lemma 1.4 then yields

$$\Delta\Phi(x) = \rho(x) - \int_{\partial B_R(0)} \rho(y)\frac{\partial}{\partial n_y}G(x,y) - G(x,y)\frac{\partial}{\partial n}\rho(y)\, d\sigma.$$

Now, both ρ and $\frac{\partial}{\partial n}\rho$ are zero on the boundary of B_R so that indeed $\Delta\Phi(x) = \rho(x)$. ■

We remark that the solution presented in Theorem 1.11 is not unique. Indeed, by adding any harmonic function Ψ, we get

$$\Delta(\Phi + \Psi)(x) = \rho(x), \qquad x \in \mathbb{R}^n.$$

Nevertheless, we get uniqueness under the additional condition that the solution of the Poisson problem has a certain behavior for large $|x|$. The solution in Theorem 1.11 satisfies in the case supp $\rho \subset B_R(0)$

$$|\Phi(x)| \leq \int_{\mathbb{R}^n} |g(|x-y|)|\, dy \|\rho\|_\infty$$

where $\|\cdot\|_\infty$ is the supremum norm on $\mathbb{R}^n$. Since

$$|x-y| = |x| \left| \frac{x}{|x|} - \frac{y}{|x|} \right| \leq \left(1 + \frac{R}{|x|}\right) |x|, \qquad \forall y \in B_R(0)$$

we get

$$\frac{g(|x-y|)}{g(|x|)} = \mathcal{O}(1), \qquad \text{for } |x| \to \infty$$

so that also

$$\left| \frac{\Phi(x)}{g(|x|)} \right| = \mathcal{O}(1), \qquad \text{for } |x| \to \infty.$$

If $n > 2$ this implies in particular that $\Phi(x) \to 0$ for $|x| \to \infty$.

Theorem 1.12. **(Uniqueness)** *For $n > 2$, the solution Φ from Theorem 1.11 is unique under the condition*

$$\Phi(x) \to 0, \qquad \textit{for } |x| \to \infty. \tag{1.12}$$

Proof: Assume Φ and $\tilde{\Phi}$ are two solutions of the Poisson problem which satisfy (1.12). Due to linearity, the difference $w = \Phi - \tilde{\Phi}$ is then a harmonic function ($\Delta w = 0$) which also satisfies (1.12). Consequently, there exists $R > 0$ such that $w(x)$ is bounded uniformly for $|x| > R$. If the support of the source ρ is contained in the ball $B_{\tilde{R}}(0)$, we get on $|x| \leq R$

$$|w(x)| = \|\rho\|_\infty \sup_{|x| \leq R} \int_{B_{\tilde{R}}(0)} G(x,y)\, dy < \infty$$

so that w is a bounded harmonic function on $\mathbb{R}^n$. According to Theorem 1.6, w is constant and since w vanishes for $|x| \to \infty$ this constant must be zero which yields uniqueness. ■

Another important property of the solution Φ given in Theorem 1.11 is its continuous dependence on ρ (the data of the problem).

Theorem 1.13. (Continuous dependence)
Let $\rho, \hat{\rho} \in C_0^2(\mathbb{R}^n)$ with supports contained in a ball of radius $R > 0$ around the origin. If $\|\cdot\|_\infty$ denotes the supremum norm in $\mathbb{R}^n$, we have for any compact subset K of $\mathbb{R}^n$

$$\sup_{x\in K} \|\Phi[\rho](x) - \Phi[\hat{\rho}](x)\| \leq C(R, K)\|\rho - \hat{\rho}\|_\infty$$

where $\Phi[\rho]$ is the solution of the problem $\Delta\Phi = \rho$ given in Theorem 1.11.

Proof: We choose $\tilde{R} \geq R$ such that $K + B_R(0)$ is contained in a ball of radius $\tilde{R}$ around the origin. Then for $x \in K$, we have

$$\begin{aligned} |\Phi[\rho](x) - \Phi[\hat{\rho}](x)| &\leq \int_{\mathbb{R}^n} |G(x,y)|\,|\rho(y) - \hat{\rho}(y)|\,dy \\ &\leq \int_{B_{\tilde{R}}(0)} |G(x,y)|\,dy\|\rho - \hat{\rho}\|_\infty \leq \int_0^{\tilde{R}} |g(r)|r^{n-1}\,dr\|\rho - \hat{\rho}\|_\infty \end{aligned}$$

■

For a general linear elliptic differential operator L with constant coefficients, a function (resp. distribution N) which satisfies the equation

$$LN = \delta_0, \qquad \text{in } \mathbb{R}^n$$

is called *fundamental solution* of L. Hence,

$$N(x) = \begin{cases} \frac{1}{\omega_1}|x| & n = 1 \\ \frac{1}{\omega_2}\ln|x| & n = 2 \\ \frac{1}{(2-n)\omega_n}|x|^{2-n} & n > 2 \end{cases}$$

is the fundamental solution of the Laplace operator. We have seen that the Poisson problem $\Delta\phi = \rho$ can be solved by *convolution* of the source ρ with the fundamental solution N

$$\phi(x) = \int_{\mathbb{R}^n} N(x-y)\rho(y)\,dy.$$

This is also true for general operators L as we shall see in Section 2.4. If we consider the Poisson problem on a more general domain

$$\begin{aligned} \Delta\phi = \rho \quad &\text{in } \Omega \\ \phi = 0 \quad &\text{on } \partial\Omega \end{aligned}$$

it would still be convenient if the solution could be represented by an integral operator acting on ρ

$$\phi(x) = \int_\Omega G(x,y)\rho(y)\,dy.$$

The kernel G of the solution operator, which can be viewed as a generalization of the fundamental solution, is called *Green's function.*

Finally, we want to give an example where the general Dirichlet problem on a ball of radius R around the origin is solved by integral operators.

Theorem 1.14. *Let $\Omega = B_R(0)$ and let g be continuous on $\partial\Omega$. Then, the function*

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial\Omega} \frac{g(y)}{|x-y|^n}\, d\sigma(y), \qquad x \in \Omega \tag{1.13}$$

satisfies $\Delta u = 0$ in Ω and for every $y \in \partial\Omega$, we have

$$\lim_{x\to y} u(x) = g(y).$$

Proof: For abbreviation, we introduce

$$K(x,y) = \frac{R^2 - |x|^2}{\omega_n R} \frac{1}{|x-y|^n}, \qquad x \in \Omega,\ y \in \partial\Omega$$

so that

$$u(x) = \int_{\partial\Omega} K(x,y) g(y)\, d\sigma(y).$$

If $x \in \Omega$, we can differentiate under the integral and find $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω. It remains to check the boundary values of u. To this end, we introduce

$$\psi(x) = \int_{\partial\Omega} K(x,y)\, d\sigma(y)$$

which is harmonic in Ω due to the remarks above. Moreover, ψ is radially symmetric. At the beginning of Section 1.2.2, we have already determined the radially symmetric solutions of Laplace's equation. Since ψ is smooth, the constant c_2 in (1.3) has to be zero which shows that ψ is, in fact, constant. More precisely, with

$$K(0,y) = \frac{R}{\omega_n} \frac{1}{|y|^n}$$

we have

$$\psi(x) = \psi(0) = \frac{R}{\omega_n} \frac{\omega_n R^{n-1}}{R^n} = 1.$$

Now, let $x_0 \in \partial\Omega$ and let $\epsilon > 0$ be given. Choose $\delta > 0$ so that $|g(y) - g(x_0)| < \epsilon$ whenever $|y - x_0| < \delta$. Then

$$\begin{aligned} |u(x) - g(x_0)| &= \left| \int_{\partial\Omega} K(x,y)(g(y) - g(x_0))\, d\sigma(y) \right| \\ &\le \int_{\{y\in\partial\Omega : |y-x_0| \le \delta\}} K(x,y) |g(y) - g(x_0)|\, d\sigma(y) \\ &\quad + \int_{\{y\in\partial\Omega : |y-x_0| > \delta\}} K(x,y) |g(y) - g(x_0)|\, d\sigma(y) \\ &\le \epsilon + \int_{\{y\in\partial\Omega : |y-x_0| > \delta\}} K(x,y) |g(y) - g(x_0)|\, d\sigma(y) \end{aligned}$$

If x is sufficiently close to x_0 (e.g. $|x - x_0| < \delta/2$) we have $|x - y| > \delta/2$ if $y \in \partial\Omega$ and $|y - x_0| > \delta$. Then

$$K(x, y) \leq \frac{R^2 - |x|^2}{\omega_n R(\delta/2)^n}$$

on that set and if M is an upper bound for g on $\partial\Omega$ we find

$$|u(x) - g(x_0)| \leq \epsilon + \frac{2M(R^2 - |x|^2)R^{n-2}}{(\delta/2)^n}.$$

For $x \to x_0$, we have in particular $|x| \to R$ so that also the second term is less than ϵ for x close enough to x_0. ■

Equation (1.13) is known as *Poisson formula.* The case $x = 0$ leads to the mean-value property presented in Theorem 1.5.

1.3.3. Variational formulation. The variational approach to elliptic PDEs is an extremely powerful technique. Although it is not as elementary as the previous examples (in the sense that quite some heavy mathematical theory is required for a rigorous treatment), we still present it at this point to stress the main ideas.

The general principle is found in many fields of physics where the unique solution of some differential equation can be interpreted as optimizer of a functional (typically an energy functional according to the principle that processes in nature use energy in an optimal way). We will see in the following, that an *energy functional* corresponding to Laplace's equation on a bounded domain Ω is given by

$$\mathscr{E}(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx.$$

Our aim is to characterize the minimizer $\bar{u}$ of $\mathscr{E}$ under the constraint that $\bar{u} = f$ on the boundary $\partial\Omega$. Hence, our class of admissible functions is given by

$$\mathcal{A}_f := \{\, u : \Omega \mapsto \mathbb{R} \,:\, u(x) = f(x) \text{ for } x \in \partial\Omega,\ \mathscr{E}(u) < \infty \,\}.$$

Similar to functionals on a finite dimensional space, a necessary condition in an optimum is that all directional derivatives vanish. A valid *direction* in the set $\mathcal{A}_f$ is a function $v : \Omega \mapsto \mathbb{R}$ which vanishes on the boundary and satisfies $\mathscr{E}(v) < \infty$, i.e. $v \in \mathcal{A}_0$. Now, if $\bar{u} \in \mathcal{A}$ is the minimum of the functional $\mathscr{E}$, then

$$\bar{u} + tv \in \mathcal{A} \qquad \forall t \in \mathbb{R}, \quad v \in \mathcal{A}_0.$$

The necessary condition for an optimum in $\bar{u}$ can be formulated as

$$\frac{d}{dt}\mathscr{E}(\bar{u} + tv)\bigg|_{t=0} = 0, \qquad \forall v \in \mathcal{A}_0. \tag{1.14}$$

Using the definition of $\mathscr{E}$, we get

$$\mathscr{E}(\bar{u}+tv) = \mathscr{E}(\bar{u}) + t\int_\Omega \nabla\bar{u}\cdot\nabla v\,dx + t^2\mathscr{E}(v)$$

so that (1.14) implies

$$\int_\Omega \nabla\bar{u}\cdot\nabla v\,dx = 0, \qquad \forall v\in\mathcal{A}_0. \tag{1.15}$$

With partial integration (under the assumption of sufficient smoothness), we deduce further

$$0 = \int_\Omega \nabla\bar{u}\cdot\nabla v\,dx = -\int_\Omega v\Delta\bar{u}\,dx + \int_{\partial\Omega} v\frac{\partial\bar{u}}{\partial n}\,d\sigma$$

and since v vanishes on $\partial\Omega$, we get

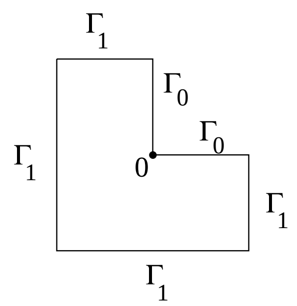
$$\int_\Omega v\Delta\bar{u}\,dx = 0 \qquad \forall v\in\mathcal{A}_0. \tag{1.16}$$

The result which allows us to use (1.16) to deduce that $\bar{u}$ satisfies Laplace's equation is referred to as *fundamental lemma of the calculus of variation*. Assuming that $\Delta\bar{u}$ is continuous and not identically zero, we just pick a point $\bar{x}\in\Omega$ such that, for example, $\Delta\bar{u}(\bar{x})>0$. Due to continuity, $\Delta\bar{u}$ is then positive on a whole neighborhood of $\bar{x}$. Picking a function $\bar{v}\geq 0$ which is compactly supported in that neighborhood, we get the contradiction

$$\int_\Omega \bar{v}\Delta\bar{u}\,dx > 0$$

so that indeed $\Delta\bar{u}=0$.

We stress that all arguments presented here are more or less formal. In particular, the conditions on the boundary value f in conjunction with the shape of the boundary $\partial\Omega$ could imply $\mathcal{A}=\emptyset$ (if no function u of finite energy $\mathscr{E}(u)$ can have these boundary values). Consequently, restrictions have to be put on f and $\partial\Omega$. Also, the question of existence of a minimizer $\bar{u}$ in $\mathcal{A}$ has yet to be answered. Finally, most arguments in the outline above required some smoothness of the minimizer (like $C^2(\bar{\Omega})$) – again a topic which has to be clarified. In fact, it turns out that $C^2(\bar{\Omega})$ cannot always be achieved. As example we mention the following geometry.



Using polar coordinates

$$x = r\cos\phi, \qquad y = r\sin\phi$$

we find that

$$u(r,\phi) = r^{\frac{2}{3}} \sin(2\phi - \pi)$$

is a solution of Laplace's equation with smooth boundary data (in fact, $u \equiv 0$ on Γ_0). Nevertheless, the first derivatives of u are unbounded in $r = 0$ so that $u \notin C^1(\bar{\Omega})$.

Due to such problems, it turns out to be useful to start with a weaker solution concept than in the classical approach. We have already seen that (1.16) implies $\Delta\bar{u} = 0$ if $\bar{u}$ is smooth enough. In that sense, (1.16) is a generalized solution concept since it requires less regularity of $\bar{u}$ but reduces to the classical solution if enough regularity is there. In view of these considerations, relation (1.15) is called *weak form* of Laplace's equation and a solution $\bar{u}$ of the weak form is called *weak solution* (as opposed to *classical solutions* which obey $\Delta\bar{u} = 0$ point wise). The problems touched here are common in the modern theory of PDEs. It is often easy to find some sort of weak solution to an equation but relatively hard to show that the weak solution is in fact a classical solution.

1.4. Fundamental questions

Given some elliptic PDE, we try to answer the following questions:

1) Does a solution exist?
2) Is the solution unique?
3) Does the solution depend continuously on the data of the problem?
4) How can the solution be approximated numerically?

A problem for which questions one to three have positive answers are called *well posed (in the sense of Hadamard)*. Otherwise the problem is said to be *ill posed*. Note, that well–posedness is equivalent to the existence of a continuous solution operator (which maps the given data to the solution).

To answer the above questions, we will first adopt a classical point of view (smooth functions, finite difference approximations). Then, we will switch over to a weak formulation (Sobolev spaces, finite element methods) which is strongly related to the variational approach presented in the previous section.

1.5. Exercises

Exercise 1.1. Show that the *biharmonic operator* $\Delta^2 = \Delta\Delta$ is elliptic.

Exercise 1.2. Show that the Laplace operator in general curvilinear coordinates $y_1, \ldots, y_n$ has the form

$$\Delta_x u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i} \left(g^{ij} \sqrt{g} \frac{\partial u}{\partial y_j} \right)$$

where g^{ij} is the metric tensor

$$g^{ij} = \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k}, \qquad g = 1/\det(g^{ij}).$$

Exercise 1.3. Verify that the Laplacian takes the following form in polar coordinates

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

Exercise 1.4. Consider a linear, second order elliptic equation with constant coefficients

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial y_i} + cu = 0, \qquad y \in \Omega.$$

Show that by some suitable transformation of variables $y = y(x)$, the equation can be reduced to

$$\Delta_x u + \sum_{i=1}^{n} \tilde{b}_i \frac{\partial u}{\partial x_i} + cu = 0.$$

Is that also possible, if a_{ij} are continuous functions of y?

Exercise 1.5. Let Ω be the unit disk in $\mathbb{R}^2$ without the origin (i.e. the boundary of Ω is the unit circle S^1 and the origin). Show that the following boundary value problem has *no* solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega \\ u &= 1 \qquad \text{on } S^1 \\ u &= 0 \qquad \text{in } x = 0 \end{aligned}$$

(Hint: Show first that if there is a solution, then there is also a radially symmetric solution.)

Exercise 1.6. Let Ω be a domain in $\mathbb{R}^n$. For abbreviation we set $L := -\Delta$ for the negative Laplacian. A function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is called *subharmonic* if $Lu \leq 0$ in Ω and *superharmonic* if $Lu \geq 0$.

a) Let u be subharmonic in Ω. Show that for any $x \in \Omega$ and any $R > 0$ such that $\bar{B}_R(x) \subset \Omega$, we have

$$u(x) \leq \frac{1}{\operatorname{vol}(B_R(x))} \int_{B_R(x)} u(y)\, dy.$$

(Hint: Similar to the proof of Theorem 1.5.)

b) Let u be subharmonic in Ω and let $A = \sup_{x \in \bar{\Omega}} u(x)$. If Ω is connected, then either $u = A$ throughout Ω or $u < A$ in Ω.

Exercise 1.7.

a) Derive the formula for the surface area ω_n of the unit sphere in $\mathbb{R}^n$. (Hint: integrate $\exp(-|x|^2)$ in Cartesian and spherical coordinates.)

b) Show that the volume of a ball of radius R is $\omega_n R^n/n$.

Exercise 1.8. Let u be a continuous function on $B_{\tilde{R}}(0)$.

a) By considering the difference quotient, calculate the derivative

$$\frac{d}{dr}\int_{B_r(0)} u(y)\,dy\bigg|_{r=R}, \qquad 0 < R < \tilde{R}.$$

b) Show that the mean value property over balls

$$u(0) = \frac{1}{|B_R(0)|}\int_{B_R(0)} u(y)\,dy, \qquad \forall\ 0 < R < \tilde{R}$$

implies the mean value property over spheres

$$u(0) = \frac{1}{|\partial B_R(0)|}\int_{\partial B_R(0)} u(y)\,d\sigma(y), \qquad \forall\ 0 < R < \tilde{R}.$$

Exercise 1.9. Show that, if $n \geq 3$, subharmonic functions on $\Omega = \mathbb{R}^n$ (as defined in Exercise 1.6) can be bounded without being constant. What about the case $n = 2$?

Exercise 1.10. Let $\Omega \subset \mathbb{R}^2$ be the unit square and let $b(x,y) = (1 - |x - 4/7|)(1 - |y - 3/8|)$. Use separation of variables (Section 1.3.1) and any math–package (maple, matlab, etc.) to find an approximate solution of $\Delta u = 0$ in Ω and $u = b$ on $\partial\Omega$.

a) Compare the Fourier–sine approximations with the exact boundary values for $M = 5$ and $M = 10$ at the northern and eastern boundary (plots!).
b) Give an isoline plot of the solution for $M = 10$.
c) Give an estimate for the maximal error of the approximation ($M = 10$) in Ω.

Exercise 1.11. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be harmonic in the domain $\Omega \subset \mathbb{R}^n$. Show that for any $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) \geq d$, the following estimate holds for the derivative

$$\left|\frac{\partial u}{\partial x_i}(x)\right| \leq \frac{n}{d}\sup_{y \in \partial B_d(x)} |u(y)|.$$

Hint: First, show a mean value property for the derivatives of u.

Exercise 1.12. Investigate the relation between harmonic and holomorphic (i.e. complex differentiable) functions.

a) Show: if f is holomorphic, its real and imaginary part is harmonic.
b) What are the counterparts of the mean value property (Theorem 1.5), the maximum principle and Theorem 1.6 ? Find other similarities...