CHAPTER 2

Classical solutions

In this chapter we investigate second–order elliptic operators of the form

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu.$$

The following assumptions are made throughout:

- Ω is an open subset of $\mathbb{R}^n$,
- a_{ij}, b_i and c are continuous on $\bar{\Omega}$,
- the matrix $A = (a_{ij})$ is symmetric and positive definite on $\bar{\Omega}$.

The problems we want to solve are of the form

$$Lu = f \qquad \text{in } \Omega \tag{2.1}$$

with $f \in C(\bar{\Omega})$. On the boundary $\partial\Omega$ we require the so called Dirichlet condition

$$u = g \qquad \text{on } \partial\Omega \tag{2.2}$$

where g is a continuous function on the boundary. In the theory of classical solutions, all derivatives in (2.1) are assumed to be defined in the classical sense. Also, for (2.2) to be meaningful, u should not have jumps at the boundary (otherwise any solution of (2.1) could be converted into a function satisfying the Dirichlet condition just by defining $u = g$ on $\partial\Omega$). Thus, our second requirement is $u \in C(\bar{\Omega})$.

Definition 2.1. *Any function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ which satisfies* (2.1) *and* (2.2) *is called* classical solution *of the Dirichlet problem.*

Beside the Dirichlet condition, other possible boundary conditions are the so called *Neumann conditions*

$$n \cdot (A\nabla u) = g, \qquad \text{on } \partial\Omega \tag{2.3}$$

or mixtures between Dirichlet and Neumann conditions

$$\alpha u + \beta n \cdot (A\nabla u) = g, \qquad \text{on } \partial\Omega. \tag{2.4}$$

where n denotes the outer unit normal vector to Ω. The theory of classical solutions for the Neumann problem will not be considered here in detail. We remark that the case $A = \mathrm{id}$ gives rise to the negative Laplacian $L = -\Delta$ for which the Neumann condition is a condition on the normal derivative $\frac{\partial u}{\partial n} = g$.

To answer the fundamental questions raised in Section 1.4 the most important tool will be the *maximum principle.* It can easily be used to show convergence and continuous dependence on the data. Also, the existence proof for the Laplace Dirichlet problem uses arguments which are based on the maximum principle. It even appears in a discrete form in stability proofs for finite difference schemes which are introduced as a numerical solution technique in Section 2.7.2.

2.1. Maximum Principles

2.1.1. The weak maximum principle. We recall that a necessary condition for $u \in C^2(\Omega)$ to have a maximum in $x_0 \in \Omega$ is

$$\frac{\partial u}{\partial x_i}(x_0) = 0, \qquad i = 1, \ldots, n.$$

Moreover, in a maximum the matrix of second derivatives is negative semi definite, i.e.

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)\xi_i\xi_j \leq 0, \qquad \forall \xi \in \mathbb{R}^n. \tag{2.5}$$

Since $A(x_0) = (a_{ij}(x_0))$ is symmetric and positive definite, the square root $S = \sqrt{A(x_0)}$ can be defined. It is a positive definite, symmetric matrix which satisfies $A(x_0) = S^2$. Due to (2.5), we get

$$-\sum_{i,j=1}^{n} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) = -\sum_{k=1}^{n}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) S_{ik}S_{jk} \geq 0.$$

Hence, if the zeroth-order coefficient c of L vanishes on Ω, then u can only have a maximum in $x_0 \in \Omega$ if $(Lu)(x_0) \geq 0$. This observation is the basis for the weak maximum principle.

Theorem 2.2. (weak maximum principle)
Assume that $Lu \leq 0$ in a bounded domain Ω and that $c \equiv 0$ in Ω. Then, the maximum of u is achieved on the boundary of Ω. If $Lu < 0$, the maximum cannot be achieved in Ω.

Proof: The case $Lu < 0$ contradicts the existence of a maximum in Ω due to our considerations above. Nevertheless, u achieves its maximum on $\bar{\Omega}$ due to continuity. Correspondingly, the maximum must be located on $\partial\Omega$.

The more general case $Lu \leq 0$ is reduced to the previous one by a perturbation argument. Introducing the function

$$u_\epsilon(x) = u(x) + \epsilon \exp(\gamma x_1)$$

we find

$$Lu_\epsilon(x) = Lu(x) - \epsilon\left(\gamma^2 a_{11}(x) - \gamma b_1(x)\right)\exp(\gamma x_1).$$

Choosing γ large enough so that $\gamma^2 a_{11} - \gamma b_1 > 0$ throughout Ω (which is possible since $a_{11} = e_1 \cdot (Ae_1) > 0$ is continuous in $\bar{\Omega}$), we get $Lu_\epsilon < 0$ on Ω for any $\epsilon > 0$. Thus

$$\max_{\bar{\Omega}} u_\epsilon = \max_{\partial\Omega} u_\epsilon .$$

Since $u_\epsilon \to u$ for $\epsilon \to 0$, we also have

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u .$$

■

We note that the adjective *weak* refers to the fact that the principle does not exclude that the maximum is also attained inside of Ω. In the *strong maximum principle* this possibility is essentially removed. As usual, the maximum principle 2.2 (and also the following results) can always be formulated as minimum principle by changing the sign of u. More precisely, if $Lu \geq 0$ throughout Ω and $c \equiv 0$ then the minimum of u is attained on the boundary. We also remark that the restriction $c \equiv 0$ can be relaxed if u has a definite sign on Ω. If, for example, $u \geq 0$ in Ω then we still find $Lu(x_0) \geq 0$ in maxima provided $c \geq 0$. Consequently, $Lu \leq 0$ implies that maxima are achieved on the boundary also in that case. If no information on the sign of u is available, we can still use these considerations to get the following weaker result

Corollary 2.3. *Let Ω be bounded and assume $c \geq 0$ in Ω and $Lu \leq 0$. Then*

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$$

where $u^+ = \max(u, 0)$. In particular, if $Lu = 0$ in Ω, then

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$$

Proof: If $u \leq 0$ everywhere, the corollary is trivially true since $u^+ = 0$ in that case. Let us therefore turn to the case where

$$\Omega^+ := \{ x \in \Omega \, : \, u(x) > 0 \}$$

is not empty. Clearly, on Ω^+ we have

$$\tilde{L}u := Lu - cu \leq 0.$$

Applying the weak maximum principle to $\tilde{L}$ on Ω^+, we get

$$\max_{\bar{\Omega}} u = \max_{\bar{\Omega}^+} u = \max_{\bar{\Omega}^+} u^+ = \max_{\partial\Omega^+} u^+ .$$

The maximum on $\partial\Omega^+$ cannot be achieved at a point $x_0 \in \partial\Omega^+ \cap \Omega$ since $u(x_0) = 0$ but $u > 0$ on Ω^+. Hence,

$$\max_{\partial\Omega^+} u^+ = \max_{\partial\Omega} u^+$$

which completes the first part of the proof. If $Lu = 0$, we simultaneously have $Lu \leq 0$ and $Lu \geq 0$ throughout Ω. Consequently,

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+ \leq \max_{\partial\Omega} |u|.$$

With $u^- = \min(u, 0)$ we obtain in the same way

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u^- \geq -\max_{\partial\Omega} |u|$$

so that the result follows. ■

With the previous result, we get uniqueness for the Dirichlet problem as a simple consequence of the maximum principle.

Corollary 2.4. (Uniqueness)
Let Ω be bounded and $c \geq 0$. Then, the Dirichlet problem

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

admits at most one classical solution.

Proof: Assume u, v are two solutions and consider $w = u - v$. Then, $Lw = 0$ in Ω and $w = 0$ on the boundary. Using Corollary 2.3, we get

$$\max_{\bar{\Omega}} |w| = \max_{\partial\Omega} |w| = 0$$

so that really $u = v$ in $\bar{\Omega}$. ■

Finally, let us mention an important monotonicity property of the solution operator of the Dirichlet problem. By combining L with the boundary operator

$$B : C(\bar{\Omega}) \mapsto C(\partial\Omega), \qquad Bu := u|_{\partial\Omega}$$

into the operator $\mathcal{L} = \binom{L}{B}$, we can formulate the Dirichlet problem as

$$\mathcal{L}u = d \tag{2.6}$$

if d denotes the *data* vector of the problem $d = \binom{f}{g}$. To get a concise formulation of the following result, we introduce the order relation $\geq$ on pairs by component wise application, i.e.

$$d_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \geq \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = d_2 \quad \Leftrightarrow \quad f_1 \geq f_2 \quad \text{and} \quad g_1 \geq g_2.$$

Corollary 2.5. *Let Ω be bounded and $c \geq 0$. Then*

$$\mathcal{L}u \geq \mathcal{L}v \quad \Longrightarrow \quad u \geq v,$$

or equivalently,

$$\mathcal{L}u \geq 0 \quad \Longrightarrow \quad u \geq 0.$$

Proof: Again, we set $w = v - u$ and find $Lw \leq 0$ in Ω and $w \leq 0$ on the boundary. Hence, $w^+ = 0$ and Corollary 2.3 yields $w \leq 0$ throughout Ω which gives $u \geq v$. ∎

For a moment, let us assume the existence of the inverse operator $\mathcal{L}^{-1}$ which maps the vector $d = \binom{f}{g}$ to the solution of the Dirichlet problem with the corresponding data (thus LB^{-1} is the *solution operator* of the Dirichlet problem). Plugging the relation $u = \mathcal{L}^{-1}d$ into the result of Corollary 2.5, we find

$$d \geq 0 \qquad \Longrightarrow \qquad \mathcal{L}^{-1}d \geq 0$$

so that $\mathcal{L}^{-1}$ is a *positive operator.* We remark that positive operators are interesting objects in their own right. They show such a rich structure that investigations independent of elliptic PDEs are rewarding and instructive. In view of this important property of the solution operator, it is not surprising, that in the numerics section elements form the theory of positive matrices are frequently used (positive matrices are the finite dimensional versions of positive operators).

2.1.2. The strong maximum principle. According to Theorem 2.2, the maximum of u is not achieved in Ω if $Lu < 0$. For the case $Lu \leq 0$, the weak maximum principle just assures that the maximum is taken on the boundary but it does not rule out the possibility that maximal values are also found inside Ω. The strong maximum principle clarifies this topic.

Theorem 2.6. (strong maximum principle)
Let Ω be a connected open subset of $\mathbb{R}^n$, assume $Lu \leq 0$ in Ω and assume that u is not constant. If $c \equiv 0$, then u does not achieve its maximum in the interior of Ω. If $c \geq 0$, u cannot achieve a non–positive maximum in the interior. Regardless of the sign of c, u cannot be zero at an interior maximum.

Before we proof the theorem, we first consider a preparatory result which is interesting in its own right.

Lemma 2.7. *Assume $Lu \leq 0$ and let x_0 be a point on $\partial\Omega$ such that $u(x_0) > u(x)$ for every $x \in \Omega$. Also assume that there exists $y \in \Omega$, $R > 0$ such that $B_R(y) \subset \Omega$ and $x_0 \in \partial B_R(y)$. Finally, the derivative of u in the direction of $n(x_0) = (x_0 - y)/R$ is supposed to exist in x_0. If either*

1) *$c = 0$, or*
2) *$c \geq 0$ and $u(x_0) \geq 0$, or*
3) *$u(x_0) = 0$*

then the directional derivative $\frac{\partial u}{\partial n}(x_0)$ is strictly positive.

Proof: Since $u(x_0) > u(x)$ for all $x \in \Omega$, we already know

$$\frac{\partial u}{\partial n}(x_0) \geq 0.$$

To get strict inequality, we again use a perturbation argument. The perturbation function

$$v(x) = \exp(-\alpha|x-y|^2) - \exp(-\alpha R^2), \qquad \alpha > 0$$

is chosen in such a way that it vanishes on $\partial B_R(y)$ and is positive inside the shell

$$S := \left\{ x \,:\, \frac{R}{2} < |x-y| < R \right\}.$$

Applying the differential operator, we get

$$\begin{aligned} Lv(x) = &-\exp(-\alpha|x-y|^2)\Bigg(4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x)(x_i-y_i)(x_j-y_j) \\ &+ 2\alpha \sum_{i=1}^{n} (a_{ii}(x) + b_i(x)(x_i-y_i)) - c(x)\left(1-\exp(-\alpha(R^2-|x-y|^2))\right)\Bigg). \end{aligned}$$

For large enough α and $x \in S$, the term in the bracket will be positive so that we can assume

$$Lv \le 0 \qquad \text{in } S. \tag{2.7}$$

Due to continuity of u and compactness, we get

$$\max_{|x-y|=R/2}(u(x) - u(x_0)) < 0.$$

In particular, we can choose $\epsilon > 0$ so that even

$$\max_{|x-y|=R/2}(u(x) - u(x_0) + \epsilon v(x)) < 0.$$

Since, by construction, $v(x) = 0$ for $x \in \partial B_R(y)$, we get altogether

$$u(x) - u(x_0) + \epsilon v(x) \le 0, \qquad x \in \partial S.$$

Using (2.7) and the assumptions 1), 2) or 3), we find

$$L(u(x) - u(x_0) + \epsilon v(x)) \le -u(x_0)c \le 0.$$

For $c \ge 0$, the weak maximum principle (Theorem 2.2) yields

$$u(x) - u(x_0) + \epsilon v(x) \le 0, \qquad x \in S.$$

Consequently,

$$\begin{aligned} \frac{\partial u}{\partial n}(x_0) &= \lim_{h \searrow 0} \frac{u(x_0) - u(x_0 - hn(x_0))}{h} \\ &\ge \epsilon \lim_{h \searrow 0} \frac{v(x_0 - hn(x_0)) - v(x_0)}{h} = -\epsilon \frac{\partial v}{\partial n}(x_0) \end{aligned}$$

and since

$$\nabla v(x_0) = -2\alpha \exp(-\alpha R^2)(x_0 - y), \qquad n(x_0) = (x_0 - y)/R$$

we get

$$-\epsilon\frac{\partial v}{\partial n}(x_0) = 2\epsilon\alpha R\exp(-\alpha R^2) > 0$$

which completes cases 1) and 2). The case $u(x_0) = 0$ is reduced to the previous ones by considering the modified differential operator

$$\tilde{L} = L - c^-, \qquad c^- = \min(c, 0).$$

Since $u(x) < u(x_0) = 0$ and $c^- \leq 0$, we get

$$\tilde{L}u = Lu - c^- u \leq Lu \leq 0.$$

Moreover, the zeroth order part of $\tilde{L}$ is given by the continuous function $c^+ = c - c^- \geq 0$ so that our previous result can be applied to $\tilde{L}$. ■

Now, we are in the position to prove the strong maximum principle.

Proof: (of Theorem 2.6)
We assume that u is not constant and assumes its maximum M at some point in Ω. Then, $A := \{x : u(x) = M\}$ is a non empty set which is relatively closed in Ω. If we show, that A is at the same time open, we get that A is the whole set Ω (due to connectedness), which contradicts the assumption that u is not constant. To show that A is open, we use a contradiction argument. Assume there is a point $z \in A$, so that each ball around z contains points where u is less than M. In particular, we find a point y with $u(y) < M$ whose distance to the boundary is larger than its distance to z. consequently, the largest ball $B_R(y)$ which is completely contained in $\{x : u(x) < M\}$ touches A in some point x_0. Applying the previous lemma to $B_R(y)$, we conclude that ∇u is non zero at x_0 contradicting the assumption that u assumes its maximum there. ■

2.2. Continuous dependence on the data

The maximum principle can be used to obtain point wise estimates for solutions of $Lu = f$ in bounded domains. To state these bounds, we introduce the following quantities

$$\lambda(x) = \min_{\xi\in S^{n-1}} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j, \qquad \beta = \max_{\bar{\Omega}} \frac{|b|}{\lambda}.$$

We have the following result.

Theorem 2.8. *Assume that Ω is bounded and contained in the strip between two parallel planes of distance d. Assume also that $c \geq 0$. If $Lu = f$ in Ω and $u = g$ on $\partial\Omega$, then*

$$\max_{\bar{\Omega}} |u| \leq \max_{\partial\Omega} |g| + C \max_{\bar{\Omega}} \frac{|f|}{\lambda}.$$

Here, $C = \exp((\beta + 1)d) - 1$.

Proof: We shall assume $Lu \le f$ and prove that

$$\max_{\bar\Omega} |u| \le \max_{\partial\Omega} u^+ + C \max_{\bar\Omega} \frac{f^+}{\lambda}.$$

The theorem follows by applying the result both to u and $-u$. Since λ and β do not change under rotations of the coordinate system, we may assume that Ω is contained in the strip $0 < x_1 < d$. We set $L_0 = L - c$. Then

$$L_0 \exp(\alpha x_1) = (\alpha^2 a_{11}(x) + \alpha b_1(x)) \exp(\alpha x_1).$$

Since $a_{11} = e_1 \cdot (Ae_1) \ge \lambda$ and $b_1 \ge -\lambda\beta$, we obtain

$$L_0 \exp(\alpha x_1) \ge \lambda(x)\alpha(\alpha - \beta) \exp(\alpha x_1).$$

If we choose α such that $\alpha - \beta > 1$, then $\alpha \ge 1$ and $\exp(\alpha x_1) \ge 1$ so that $L_0 \exp(\alpha x_1) \ge \lambda(x)$. Setting

$$v = \max_{\partial\Omega} u^+ + (\exp(\alpha x_1) - 1) \max_{\bar\Omega} \frac{f^+}{\lambda}$$

we find $v \ge 0$ and thus also $cv \ge 0$ in Ω and $u \le v$ on $\partial\Omega$. Using our previous result $L_0 \exp(\alpha x_1) \ge \lambda(x)$, we get

$$Lv = L_0 v + cv \ge \lambda \max_{\bar\Omega} \frac{f^+}{\lambda} \ge f^+$$

so that $L(u - v) \le f - f^+ = f^- \le 0$. An application of Corollary 2.5 yields

$$u \le v \le \max_{\partial\Omega} u^+ + (\exp(\alpha d) - 1) \max_{\bar\Omega} \frac{f^+}{\lambda}$$

completes the proof of the theorem. ■

We remark that the uniqueness result (Corollary 2.3) can be viewed as a special case of continuous dependence. Indeed, if u, v are two solutions of the Dirichlet problem, then the difference w solves the same problem with zero data and hence $w = 0$ according to the theorem above.

In terms of the combined interior and boundary operator $\mathcal{L} = \binom{L}{B}$, the result is expressed by

$$\|u\|_\infty \le \|Bu\|_\infty + C \left\| \frac{Lu}{\lambda} \right\| \le \left(1 + \frac{C}{\min_{\bar\Omega} \lambda}\right) \max\{\|Bu\|_\infty, \|Lu\|_\infty\}.$$

With the norm

$$\|d\|_\infty = \max\{\|Bu\|_\infty, \|Lu\|_\infty\}, \qquad d = \begin{pmatrix} f \\ g \end{pmatrix}$$

for data vectors, we get the concise formulation

$$\|u\|_\infty \le \tilde{C} \|\mathcal{L}u\|_\infty. \tag{2.8}$$

If the inverse operator exists, this implies the boundedness property

$$\|\mathcal{L}^{-1} d\|_\infty \le \tilde{C} \|d\|_\infty$$

(just set $u = \mathcal{L}^{-1}d$ in (2.8)). If this holds for all data vectors, we get for the operator norm $\|\mathcal{L}^{-1}\|_\infty \le \tilde{C}$. Consequently, continuous dependence of the solution just describes continuity of the inverse operator. Please note that the maximum norm $\|\cdot\|_\infty$ comes into play through the maximum principle and is very natural in this context.

2.3. Existence for the Dirichlet Laplace problem

We will show the existence of a classical solution to the Laplace problem with Dirichlet boundary conditions $g \in C(\partial\Omega)$.

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{2.9}$$

The admissible class of domains is quite general.

Definition 2.9. *Let Ω be an open subset of $\mathbb{R}^n$. We say that Ω has the* exterior sphere property *if to each point $x \in \partial\Omega$ there exists a ball $B \subset \bar{\Omega}^c$ such that $\bar{\Omega} \cap \bar{B} = \{x\}$.*

As examples we mention convex sets or sets with C^2 boundaries.

Theorem 2.10. *Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain with the exterior sphere property. Then, the problem* (2.9) *admits a unique solution.*

In order to prove Theorem (2.10), we need the concept of subharmonic functions.

Definition 2.11. *A function $u \in C(\bar{\Omega})$ is called* subharmonic *if for every ball B with $\bar{B} \subset \Omega$ and every function $h \in C(\bar{B})$ which is harmonic in B and satisfies $h \ge u$ on ∂B, we have $h \ge u$ throughout B.*
A subsolution *of the Dirichlet problem is a subharmonic function on Ω which satisfies $u \le g$ on $\partial\Omega$.*
Exchanging $\le$ by $\ge$, we similarly define the notions superhamonic *and* supersolution.

Simple examples of super- and subsolutions are the constant functions

$$\text{Max} := \max_{\partial\Omega} g, \qquad \text{Min} := \min_{\partial\Omega} g.$$

As already indicated by the names, we have that any subsolution for the Dirichlet problem is less than or equal to every supersolution. Indeed, if v is a subsolution and w a supersolution, then $v - w$ is subharmonic (Exercise 2.1) and $v - w \le 0$ on $\partial\Omega$. Using the maximum principle for subharmonic functions (Exercise 2.1), we get either $v = w$ throughout Ω or $v < w$ in Ω. Consequently, subsolutions are bounded from above by any supersolution, and thus in particular by the constant $\text{Max} = \max_{\partial\Omega} g$. Also, we can assume that subsolutions are bounded from below by going over form v to $\max(v, \text{Min})$ which is again a subsolution (see Exercise 2.1).

Our candidate for the solution of the Dirichlet problem is now the supremum of all subsolutions.

$$u(x) = \sup_{v \in \mathrm{Sb}(g)} v(x), \qquad x \in \Omega \tag{2.10}$$

where $\mathrm{Sb}(g)$ denotes the set of all subsolutions to the Dirichlet problem with boundary values g. Due to our considerations above, u is well defined and we have

$$\mathrm{Min} \le u \le \mathrm{Max}.$$

The advantage in the approach (2.10) is that differentiability considerations do not enter in the construction which simplifies the further treatment.

An important tool to prove that u is really the solution is the *harmonic lifting* of a subharmonic function v with respect to some ball $B \subset \Omega$. The idea is to enlarge v on the ball B by going over to the harmonic function $\bar{v}$ on B which has the same values at the boundary ∂B as the original function v. This function exists due to the Poisson formula (1.13)

$$V(x) = \begin{cases} \bar{v}(x) & x \in B \\ v(x) & x \in \Omega \backslash B \end{cases}$$

Since v is subharmonic and $v = \bar{v}$ on ∂B, we have $v \le \bar{v}$ in B so that $v \le V$ in Ω, i.e. the harmonic lifting V is closer to the supremum over all subsolutions than v, provided V is again a subsolution.

Lemma 2.12. *Let v be a subharmonic function on Ω and B a ball whose closure is contained in Ω. Then, the harmonic lifting V of v with respect to B is again subharmonic.*

Let $\Theta \subset \Omega$ be any ball and let h be a harmonic function in Θ which satisfies $V \le h$ on $\partial\Theta$. We have to show that then also $V \le h$ throughout Θ. Since $v \le V$ and v is subharmonic, we get from the definition that $v \le h$ in B. Hence, at least for the points in $\bar{\Theta} \backslash B$ we have $V = v \le h$. To finish the proof, we just have to consider $B \cap \Theta$. Here, V is, by construction, harmonic so that $V \le h$ follows from the corresponding relation at the boundary of $B \cap \Theta$ (due to the maximum principle for harmonic functions). By definition, we have $V \le h$ on $\partial\Theta$ and, because of the previous argument, $V \le h$ on $\partial B \cap \bar{\Theta}$. Since the boundary of $B \cap \Theta$ is contained in the union of $\partial B \cap \bar{\Theta}$ and $\partial\Theta \cap \bar{B}$, the proof is complete. ∎

Finally, we need a result on convergence of sequences of harmonic functions.

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let $(f_m)_{m \in \mathbb{N}}$ be a sequence of harmonic functions on Ω which is uniformly bounded, i.e. $|f_m(x)| \le M$ for every $x \in \Omega$ and $m \in \mathbb{N}$. Then, $(f_m)_{m \in \mathbb{N}}$ has a subsequence which converges to a harmonic function on Ω, uniformly on compact subsets of Ω.*

Proof: Let

$$\Omega_k := \left\{ x \in \Omega \, : \, |x| \le k, \qquad \operatorname{dist}(x, \partial\Omega) \ge \frac{1}{k} \right\}.$$

Then, Ω_k is compact and $\Omega = \cup_{k=1}^{\infty} \Omega_k$. Using Exercise 1.11, we find locally uniform bounds for the derivatives of all members of the sequence

$$|\nabla f_m(x)| \le nkM, \qquad \forall x \in \Omega_k.$$

Thus, $(f_m)_{m\in\mathbb{N}}$ is equicontinuous on Ω_k. By the Arzela-Ascoli theorem, we can extract a uniformly convergent subsequence on each Ω_k and using a diagonal argument, we obtain a subsequence which converges on Ω uniformly on each Ω_k. Due to Corollary 1.10, the limit of a uniformly convergent sequence of harmonic functions is again harmonic, which completes the proof. ∎

Now, we have all tools at hand to show that (2.10) is the solution of the Dirichlet problem (2.9).

Proof: (of Theorem 2.10)
First, we check that u defined in (2.10) is harmonic. To this end, we pick $x \in \Omega$ and $(v_m)_m \in \mathbb{N} \subset \mathrm{Sb}(g)$ such that $v_m(x) \to u(x)$ for $m \to \infty$. If B is a ball around x whose closure is contained in Ω, we can go over to the lifted sequence $(V_m)_{m\in\mathbb{N}}$ with respect to B. Since $v_m \le V_m$ we also have $V_m(x) \to u(x)$ since $u(x)$ is the supremum over all subsolutions at that point. Using Theorem 2.13, V_m has a convergent subsequence on B (again denoted $(V_m)_{m\in\mathbb{N}}$ to avoid sub indices), which converges uniformly on B to some harmonic function V. Since $V_m \le u$ for all m, we also have $V \le u$. We now show, that even $V = u$, which proves that u is equal to a harmonic function in a neighborhood of x and thus is harmonic.

To get a contradiction to $V = u$, we assume $V(y) < u(y)$ for some $y \in B$. Then, there exists (by definition of u as the supremum over subharmonic functions) a function $\phi \in \mathrm{Sb}(g)$, such that $V(y) < \phi(y) \le u(y)$. Let then V_m^{ϕ} be the harmonic lifting of $\max\{\phi, V_m\}$ with respect to B. As before, a subsequence of V_m^{ϕ} converges uniformly to some harmonic function V^{ϕ} in B. By construction, we have

$$V \le V^{\phi} \quad \text{in } B, \qquad V(y) < V^{\phi}(y)$$

so that certainly $\sup_{\bar{B}}(V - V^{\phi}) \le 0$. Since at x all our sequences converge to the supremum $u(x)$, we even have

$$V(x) = V^{\phi}(x), \qquad \sup_{\bar{B}}(V - V^{\phi}) = 0.$$

Since the difference $V - V^{\phi}$ is harmonic in B, we can apply the strong maximum principle which gives either

- $V - V^{\phi} = \sup_{\bar{B}}(V - V^{\phi}) = 0$, which contradicts $V(y) < V^{\phi}(y)$, or
- $V - V^{\phi} < \sup_{\bar{B}}(V - V^{\phi}) = 0$, which contradicts $V(x) = V^{\phi}(x)$.

It remains to show, that u is actually continuous up to the boundary $\partial\Omega$ and assumes the given boundary data. Let ξ be any point on $\partial\Omega$. Due to the exterior ball property, we can find a ball $B = B_R(y)$ in $(\bar{\Omega})^c$ such that $\bar{B} \cap \bar{\Omega} = \{\xi\}$. We now define

$$w(x) := \begin{cases} \ln \frac{|x-y|}{R} & \text{if } n = 2 \\ R^{2-n} - |x-y|^{2-n} & \text{if } n \neq 2 \end{cases}$$

Then

- w is (super)harmonic in Ω,
- $w > 0$ on $\bar{\Omega}\backslash\{\xi\}$ and
- $w(\xi) = 0$.

(Any continuous function with these properties is called a *barrier* at ξ relative to Ω.) Using the barrier w at ξ, we show that $u(x) \to g(\xi)$ as $x \to \xi$. First, we choose $\epsilon > 0$ and set $M := \sup_{\partial\Omega} |g|$. Then, $\delta > 0$ and $k > 0$ are selected such that

$$|g(x) - g(\xi)| < \epsilon \qquad \text{whenever } x \in \partial\Omega \text{ and } |x-y| < \delta$$

and

$$kw(x) > 2M \qquad \text{whenever } x \in \partial\Omega \text{ and } |x-y| \geq \delta.$$

The function

$$s(x) = g(\xi) - \epsilon - kw(x)$$

then satisfiers for $x \in \partial\Omega$ the relation $s(x) < g(x)$ if $|x-\xi| < \delta$, and $s(x) < -M < g(x)$ for $|x-y| \geq \delta$. Also, s is subharmonic (since w is superharmonic), so that s is, in fact, a subsolution of the Dirichlet problem. Correspondingly,

$$S(x) = g(\xi) + \epsilon + kw(x)$$

is a supersolution and we find $s(x) \leq u(x) \leq S(x)$. Since $w(x) \to 0$ for $x \to \xi$, we find due to continuity of w a neighborhood of ξ such that $|kw| \leq \epsilon$ for $|x-\xi| \leq \delta'$. Thus, for $|x-\xi| \leq \delta'$

$$g(\xi) - 2\epsilon \leq g(\xi) - \epsilon - kw(x) \leq u(x) \leq g(\xi) + \epsilon + kw(x) \leq g(\xi) + 2\epsilon.$$

This shows that u can be continuously extended to $\partial\Omega$ and that the continuation is the prescribed function g. ■

2.4. Existence for the Dirichlet Poisson problem

The Dirichlet Laplace problem (2.9) is now extended to more general sources.

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega \end{aligned} \tag{2.11}$$

The idea how to solve (2.11) is the following: Assume, we have some function $w \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $-\Delta w = f$. We then solve the auxiliary problem

$$\begin{aligned} \Delta\phi &= 0 \qquad && \text{in } \Omega \\ \phi &= g - w \qquad && \text{on } \partial\Omega \end{aligned}$$

which we can do (under suitable assumptions on Ω and g) according to Theorem 2.10. The solution of (2.11) is then obtained with $u = \phi + w$. Thus, we only need to construct a suitable function w to solve the Dirichlet Poisson problem. For $f \in C_0^2(\Omega)$ we have already found the solution in Theorem 1.11

$$w(x) = \int_\Omega g(|x-y|)(-f(y))\,dy. \tag{2.12}$$

However, the required smoothness assumptions and the restrictions on the support seem to be too strong. In fact, the integral (2.12) makes sense already for bounded and integrable functions. We even have

Lemma 2.14. *Let Ω be a bounded domain and let f be measurable and bounded on Ω. Then, the Newtonian potential*

$$w(x) = \int_\Omega g(|x-y|) f(y)\,dy$$

satisfies $w \in C^1(\bar{\Omega})$ and for any $x \in \Omega$

$$\frac{\partial w}{\partial x_i}(x) = \int_\Omega \frac{\partial}{\partial x_i} g(|x-y|) f(y)\,dy, \qquad i = 1, \dots, n.$$

Proof: Since

$$\left|\frac{\partial}{\partial x_i} g(|x-y|)\right| = \left|\frac{1}{\omega_n}\frac{1}{|x-y|^{n-1}}\frac{x_i - y_i}{|x-y|}\right| \le \frac{1}{\omega_n}\frac{1}{|x-y|^{n-1}},$$

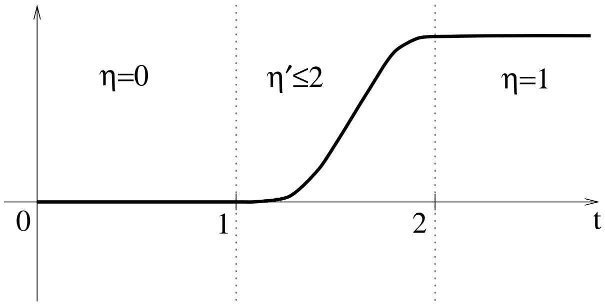
the restriction of $\partial g/\partial x_i$ on $B_1(x)$ is integrable and the one on $\Omega \backslash B_1(x)$ is bounded. Consequently,

$$\begin{aligned} \int_\Omega \left|\frac{\partial}{\partial x_i} g(|x-y|) f(y)\right| dy \le{}& \|f\|_{\mathbb{L}^\infty} \|\frac{\partial}{\partial x_i} g(|x-\cdot|)\|_{\mathbb{L}^1, B_1(x)} \\ &+ \|\frac{\partial}{\partial x_i} g(|x-\cdot|)\|_{\mathbb{L}^\infty, \Omega\backslash B_1(x)} \|f\|_{\mathbb{L}^1} \end{aligned}$$

so that

$$v(x) = \int_\Omega \frac{\partial}{\partial x_i} g(|x-y|) f(y)\,dy$$

is well defined for $x \in \bar{\Omega}$. Choosing a smooth function $\eta \in C^1(\mathbb{R})$ as in the figure below



we define $\eta_\epsilon(t) = \eta(t/\epsilon)$ so that the nonzero part starts at $t = \epsilon$ with a slope of at most $2/\epsilon$ and $\eta_\epsilon = 1$ for $t > 2\epsilon$. We use $\eta_\epsilon(|x-y|)$ to cut out the singular point in the Newtonian potential which leads to a smooth function $w_\epsilon \in C^1(\bar{\Omega})$

$$w_\epsilon(x) = \int_\Omega g(|x-y|)\eta_\epsilon(|x-y|)f(y)\,dy.$$

The difference between $\partial w_\epsilon/\partial x_i$ and v defined above is

$$v(x) - \frac{\partial w_\epsilon}{\partial x_i}(x) = \int_{|x-y|\le 2\epsilon} \frac{\partial}{\partial x_i}\left((1-\eta_\epsilon(|x-y|))g(|x-y|)\right) f(y)\,dy$$

so that

$$\left| v(x) - \frac{\partial w_\epsilon}{\partial x_i}(x) \right| \le \|f\|_{\mathbb{L}^\infty} \int_{|x-y|\le 2\epsilon} \left| \frac{\partial}{\partial x_i} g(|x-y|) \right| + \frac{2}{\epsilon}|g(|x-y|)|\,dy$$

where the bound on η_ϵ and its slope has been used. Writing the integral in spherical coordinates, we see that the Jacobian factor r^{n-1} cures the singularity of the first derivative of g, so that only a term of order ϵ remains because the range of integration (in r) is $[0, 2\epsilon]$. The second term in the integral is divided by ϵ, but also its singularity is weaker. Altogether, the integral vanishes with $\epsilon \to 0$, so that $\partial w_\epsilon/\partial x_i$ converges uniformly to v. Since also w_ϵ converges uniformly to w, the result follows. ∎

If we want to calculate the second derivatives in the same way, we face the problem, that

$$\frac{\partial^2}{\partial x_i \partial x_j} g(|x-y|) = \frac{1}{\omega_n} \frac{1}{|x-y|^n} \left(\delta_{ij} - \frac{(x_i - y_i)(x_j - y_j)}{|x|^2} \right)$$

is no longer locally integrable around x. If, however, $f(y) - f(x)$ vanishes sufficiently fast for $y \to x$, we find that

$$\left(\frac{\partial^2}{\partial x_i \partial x_j} g(|x-y|) \right) (f(y) - f(x))$$

is again integrable. More precisely, if f is locally Hölder continuous to the exponent $\alpha > 0$, i.e. for any compact $K \subset \Omega$

$$|f(y) - f(x)| < C_K |x-y|^\alpha, \qquad \forall x, y \in K$$

then

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} g(|x-y|)\right| |f(y) - f(x)| \le \frac{2C_K}{\omega_n} \frac{1}{|x-y|^{n-\alpha}}$$

which again allows a treatment as for the first derivative.

Lemma 2.15. *Let Ω be a bounded domain and let f be bounded and locally Hölder continuous on Ω. Then*

$$w(x) = \int_\Omega g(|x-y|) f(y)\, dy$$

is contained in $C^2(\Omega) \cap C^1(\bar{\Omega})$ and satisfies $\Delta w = f$.

Proof: See [**9**]. ■

Finally, we can summarize our results on the Dirichlet Poisson problem in

Theorem 2.16. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which satisfies the exterior sphere property. If f is a bounded, locally Hölder continuous function in Ω and $g \in C(\partial\Omega)$, the Dirichlet Poisson problem* (2.11) *is uniquely solvable.*

2.5. The general Dirichlet problem

The existence proof for the Dirichlet Laplace problem was essentially based on Poisson's formula and the strong maximum principle. While the maximum principle for general linear, elliptic second order operators L with $c \ge 0$ has been proven in Theorem 2.6, an extension of Poisson's formula to general cases is quite involved. Nevertheless, the basic idea is to use results for the Laplacian and extend them by functional analytic methods to other operators. The crucial point in this approach are a priori estimates similar to those obtained in Section 2.2. However, not only estimates on the function itself are required but also bounds for the derivatives in terms of the data of the problem. It turns out, in accordance with the previous section, that the coefficients of the operator and the source should be locally Hölder continuous. For a detailed investigation, we refer to [**9**]. The main result of this so called *Schauder approach* is valid for *strictly elliptic* operators L with $c \ge 0$. To explain this concept, we use our notation from Section 2.2

$$\lambda(x) = \min_{\xi \in S^{n-1}} \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j.$$

The operator L is called elliptic, if $\lambda > 0$ in Ω. If λ is even bounded away from zero, i.e. $\lambda \ge \lambda_0 > 0$ in Ω, then L is called strictly elliptic. In our original setup, we assumed that $A(x) = (a_{ij}(x))$ is continuous and positive definite in $\bar{\Omega}$ so that on bounded domains the operator is automatically strictly elliptic (due to continuity and compactness). However, the continuity and definiteness up to the boundary is only a sufficient condition for strict ellipticity.

Theorem 2.17. *Let L be strictly elliptic in a bounded domain Ω with $c \geq 0$, and let f and the coefficients of L be bounded and locally Hölder continuous with exponent $\alpha > 0$. Suppose that Ω satisfies the exterior sphere condition. Then, if g is continuous on $\partial\Omega$, the Dirichlet problem*

$$Lu = f \quad in\ \Omega, \qquad u = g \quad on\ \partial\Omega,$$

has a unique solution $u \in C(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$.

Here, $C^{2,\alpha}(\Omega)$ is the set of functions which are twice continuously differentiable in Ω and whose second derivatives are locally Hölder continuous to the exponent α. With the stronger assumption that the boundary of Ω is locally the graph of a $C^{2,\alpha}$ function (i.e. Ω is a $C^{2,\alpha}$ domain), that the boundary data are contained in $C^{2,\alpha}(\partial\Omega)$ and that all other prescribed functions are Hölder continuous up to the boundary, we get the stronger result

Theorem 2.18. *Let L be strictly elliptic in a bounded $C^{2,\alpha}$ domain Ω with $c \geq 0$, and let f and the coefficients of L belong to $C^{\alpha}(\bar{\Omega})$. If $g \in C^{2,\alpha}(\partial\Omega)$ then the Dirichlet problem*

$$Lu = f \quad in\ \Omega, \qquad u = g \quad on\ \partial\Omega,$$

has a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$.

The space $C^{2,\alpha}(\bar{\Omega})$ is a Banach space with the norm

$$\|u\|_{2,\alpha;\Omega} := \|u\|_{\infty,\Omega} + \max_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{\infty,\Omega} + \max_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{\infty,\Omega} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\left| \frac{\partial^2 u}{\partial x_i \partial x_j}(x) - \frac{\partial^2 u}{\partial x_i \partial x_j}(y) \right|}{|x-y|^{\alpha}}.$$

The following result extends the continuous dependence form Section 2.2 to this stronger space.

Theorem 2.19. *Under the assumptions of Theorem 2.18 we find for the solution u of the Dirichlet problem*

$$\|u\|_{2,\alpha;\Omega} \leq C \left(\|u\|_{\infty,\Omega} + \|g\|_{2,\alpha;\partial\Omega} + \|f\|_{0,\alpha;\Omega} \right).$$

Finally, we state that higher regularity of $\partial\Omega$, coefficients and data implies higher regularity of the solution.

Theorem 2.20. *Let Ω be a $C^{2+k,\alpha}$ domain ($k \geq 0, \alpha > 0$) and let $g \in C^{2+k,\alpha}(\partial\Omega)$. Suppose that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies $Lu = f$ in Ω, $u = g$ on $\partial\Omega$ where f and the coefficients of the strictly elliptic operator L belong to $C^{k,\alpha}(\bar{\Omega})$. Then $u \in C^{k+2,\alpha}(\bar{\Omega})$.*

2.6. Results in terms of the solution operator

We summarize the previous results as properties of the Dirichlet solution operator $\mathcal{L}$. The general assumptions are that L is a linear, second order, strictly elliptic operator on a bounded domain Ω with α–Hölder continuous coefficients and $c \geq 0$. As usual, $B : C(\bar{\Omega}) \mapsto C(\partial\Omega)$ denotes the trace operator $u \mapsto u|_{\partial\Omega}$. In the previous sections we have investigated several aspects of the Dirichlet problem $\mathcal{L}u = \binom{f}{g}$. In particular, we have studied the fundamental questions concerning existence, uniqueness and continuous dependence of the solution on the data f and g (i.e. the *well-posedness* of the problem).

In terms of the operator $\mathcal{L}$, uniqueness means injectivity and existence leads to surjectivity. Consequently, existence and uniqueness together are equivalent to the invertibility of $\mathcal{L}$. In our case,we have seen that both questions are answered positively, if $g \in C(\partial\Omega)$ and f is locally Hölder continuous, i.e. $f \in C^\alpha(\Omega)$. (We have proven this result for $L = -\Delta$ in Theorem 2.16 and stated it for general L in Theorem 2.17.) Hence,

$$\mathcal{L}^{-1} : C^\alpha(\Omega) \times C(\partial\Omega) \mapsto C^{2,\alpha}(\Omega) \cap C(\bar{\Omega}).$$

For more general data (on $C^{k+2,\alpha}$ domains with $C^{k,\alpha}(\bar{\Omega})$ coefficients and $k \geq 0$) we even have according to Theorem 2.20

$$\mathcal{L}^{-1} : C^{k,\alpha}(\bar{\Omega}) \times C^{k+2,\alpha}(\partial\Omega) \mapsto C^{k+2,\alpha}(\bar{\Omega}).$$

The third aspect of well–posedness, the continuous dependence on the data, naturally translates into a continuity property of $\mathcal{L}^{-1}$. We remark that the linearity of the problem is reflected by the linearity of $\mathcal{L}$ and $\mathcal{L}^{-1}$, so that continuity is equivalent to boundedness. As a direct consequence of the maximum principle, we obtain estimates in the maximum norm on Ω. As stated in Section 2.2, we get

$$\|\mathcal{L}^{-1}d\|_{\infty,\Omega} \leq \tilde{C}\|d\|_{\infty,\Omega}, \tag{2.13}$$

where $d = \binom{f}{g}$ is some admissible data vector from $C^\alpha(\Omega) \times C(\partial\Omega)$ (with $\|d\|_{\infty,\Omega} = \max\{\|g\|_{\infty,\partial\Omega}, \|f\|_{\infty,\Omega}\}$). Theorem 2.19 even yields the stronger estimate

$$\|\mathcal{L}^{-1}d\|_{2,\alpha;\Omega} \leq C\left(\|f\|_{0,\alpha;\Omega} + \|g\|_{2,\alpha;\partial\Omega}\right), \qquad d = \begin{pmatrix} f \\ g \end{pmatrix}$$

(where (2.13) has been used to estimate $\|u\|_{\infty,\Omega}$). Finally, the maximum principle reflects the positivity of $\mathcal{L}^{-1}$ as explained in Section 2.1.1. In a nutshell

$$\begin{array}{lcl} \text{existence+uniqueness} & \Longrightarrow & \mathcal{L}^{-1} \text{ exists,} \\ \text{linearity of the problem} & \Longrightarrow & \mathcal{L}^{-1} \text{ linear,} \\ \text{continuous dependence} & \Longrightarrow & \mathcal{L}^{-1} \text{ continuous,} \\ \text{maximum principle} & \Longrightarrow & \mathcal{L}^{-1} \text{ positive.} \end{array}$$