2.7. Finite Difference Schemes for Elliptic problems

First, we consider finite difference schemes for one–dimensional problems and develop a convergence theory. After that, we will focus on multi–dimensional problems. Our assumptions on the smoothness of the domain, the coefficients and the data of the problem are as in Theorem 2.18. More precisely,

- Ω is a bounded $C^{2,\alpha}$ domain,
- coefficients of L are in $C^\alpha(\bar{\Omega})$ for some $\alpha > 0$,
- $c \geq 0$ (to ensure existence and uniqueness)
- $g \in C^{2,\alpha}(\partial\Omega)$, $f \in C^\alpha(\bar{\Omega})$.

2.7.1. Finite Difference Schemes in 1-D. If we restrict to 1D problems, then the general, linear elliptic differential operator L of second order is defined by

$$Lu = -au'' + bu' + cu$$

with $a > 0$. In particular (see Exercise 2.7), we may take $a = 1$ and consider the Dirichlet problem on the unit interval $(0,1)$, i.e.

$$\begin{aligned} Lu = -u'' + bu' + cu = f \qquad & \text{in } \Omega = (0,1) \\ u = g \qquad & \text{on } \partial\Omega = \{0,1\} \end{aligned}$$

To formulate a finite difference scheme for this elliptic problem, we proceed in the following way:

1) we introduce a grid on Ω and the boundary $\Gamma = \partial\Omega$ by

$$\begin{aligned} \Omega_h &= \{x_i : x_i = ih, i = 1, \ldots, n-1\}, \qquad h = \frac{1}{n} \\ \Gamma_h &= \{x_0 = 0, x_n = 1\} \end{aligned}$$

and denote by u_i the discrete values $u(x_i), x_i \in \Omega_h \cup \Gamma_h$,

2) we discretize the differential operator using difference quotients, i.e.

$$\begin{aligned} u'' &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2), \quad u \in C^4 \\ u' &= \frac{u(x+h) - u(x-h)}{2h} + \mathcal{O}(h^2), \quad u \in C^3 \end{aligned}$$

3) we discretize the boundary values (which is trivial in 1D).

After these steps, we obtain a discrete version of our continuous elliptic problem $Lu = f$ at the grid points $x_i \in \Omega_h$ as follows:

(2.14)
$$\begin{gathered} -\frac{(u_{i+1} - 2u_i + u_{i-1})}{h^2} + b_i\frac{u_{i+1} - u_{i-1}}{2h} + c_iu_i = f_i, \qquad i = 1, \ldots, n-1 \\ u_0 = g(0),\ u_n = g(1) \end{gathered}$$

The coefficients in (2.14) depend on the grid point x_i, i.e. $b_i = b(x_i)$, $c_i = c(x_i)$ and $f_i = f(x_i)$. The equation is obviously linear in the three unknowns u_{i-1}, u_i and u_{i+1} and can be rewritten in the form

$$l_i u_{i-1} + d_i u_i + r_{i+1} u_{i+1} = f_i$$

with

$$l_i = -\frac{1}{h^2} - \frac{b_i}{2h}, \quad d_i = \frac{2}{h^2} + c_i, \quad r_i = -\frac{1}{h^2} + \frac{b_i}{2h}.$$

Altogether we obtain a linear system of $n+1$ equations in $n+1$ unknowns defined by

$$\begin{pmatrix} 1 & & & & \\ l_1 & d_1 & r_1 & & \\ & \ddots & \ddots & \ddots & \\ & & l_{n-1} & d_{n-1} & r_{n-1} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} g_0 \\ f_1 \\ \vdots \\ f_{n-1} \\ g_n \end{pmatrix}$$

which we abbreviate in the following as

$$\mathcal{L}_h \mathbf{u}_h = \mathbf{d}_h \tag{2.15}$$

We note that the infinite number of conditions $Lu(x) = f(x)$ for all $x \in \Omega$ is reduced to a finite number by discretization. Since the original conditions are linear in u, our discretization thus leads to a linear system of equations. Also, the operator $\mathcal{L}_h$ includes the treatment of boundary values so that it compares rather to the operator $\mathcal{L}$ introduced in Section 2.1.1 which also combines the interior operator L with the boundary operator B.

We will now investigate the solvability of the reduced system and study the relation between solutions of the original system and the discretized one. More precisely, we focus on the following questions:

1) Is the linear system always uniquely solvable?
2) What happens in the limit $h \to 0$?
3) How good is the discrete approximation?

To answer the questions, we have to perform a convergence analysis. The typical sub steps in this analysis are *consistency* and *stability* checks. In the consistency part, one tries to answer the question, how good the discretization $\mathcal{L}_h$ approximates the original operator $\mathcal{L}$. To compare the two operators, one first has to overcome the problem that that they work on different spaces. Here, we use discretization and compare the actions of the operators in the finite dimensional space. Another possibility is to suitably interpolate the discrete values and compare the operators on a function space. For discretization, we introduce an operator D_h which works on pairs of functions out of $C(\Omega) \times$

$C(\partial\Omega)$

$$D_h \begin{pmatrix} v \\ g \end{pmatrix} = \begin{pmatrix} g(x_0) \\ v(x_1) \\ \vdots \\ v(x_{n-1}) \\ g(x_n) \end{pmatrix}$$

For convenience, we use the same symbol for the discretization of functions in $C(\bar{\Omega})$ according to the rule

$$D_h u = D_h \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix}$$

where the right hand side is already defined above. To estimate the quality of approximation on the discrete level, we then introduce the *local discretization error* in a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$

$$\mathbf{l}_h(u) = \mathcal{L}_h D_h u - D_h \mathcal{L} u$$

and use it as a measure for the consistency of the approximation.

Definition 2.21.
The finite difference scheme defined by the operator $\mathcal{L}_h$ is consistent of order m in $v \in C^2(\Omega) \cap C(\bar{\Omega})$, if

$$\|\mathbf{l}_h(v)\|_{h,\infty} \le Ch^m \qquad \text{for } h \to 0$$

where

$$\|w\|_{h,\infty} = \max_{i=0,\ldots,n} |w_i|, \qquad w \in \mathbb{R}^{n+1}.$$

In contrast to the local discretization error, the actual *approximation error* is obtained by comparing the solution of the original problem with the solution of the discretization. Our aim is of course, that this error vanishes for $h \to 0$.

Definition 2.22.
The finite difference scheme defined by $\mathcal{L}_h$ is convergent of order m, if for the error $\mathbf{e}_h$ defined by

$$\mathbf{e}_h = \mathbf{u}_h - D_h u,$$

(where $\mathbf{u}_h = \mathcal{L}_h^{-1}\mathbf{d}_h$ is the approximate solution and $u = \mathcal{L}^{-1}\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ the exact solution), we have

$$\|\mathbf{e}_h\|_{h,\infty} \le Ch^m \qquad \text{for } h \to 0$$

The relation between the two notions of errors is given by the following calculation.

$$\begin{aligned} \mathcal{L}_h \mathbf{e}_h &= \mathcal{L}_h \mathbf{u}_h - \mathcal{L}_h D_h u \\ &= \mathbf{d}_h - (\mathbf{l}_h(u) + D_h \mathcal{L} u) \\ &= \mathbf{d}_h - D_h\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) - \mathbf{l}_h(u) = -\mathbf{l}_h(u) \end{aligned}$$

Hence, we have $\mathbf{e}_h = -\mathcal{L}_h^{-1}\mathbf{l}_h(u)$ and

$$\|\mathbf{e}_h\|_{h,\infty} \leq \|\mathcal{L}_h^{-1}\|_{h,\infty}\|\mathbf{l}_h(u)\|_{h,\infty}$$

But then, if $\|\mathcal{L}_h^{-1}\|_{h,\infty}$ is bounded independently of h, consistency of the scheme even means convergence.

Definition 2.23.
The scheme defined by $\mathcal{L}_h$ is called stable, if $\|\mathcal{L}_h^{-1}\|_{h,\infty} \leq K$, is bounded independent of h.

Having isolated the two sub steps *consistency* and *stability*, we immediately get

Theorem 2.24. *Let the scheme defined by $\mathcal{L}_h$ be stable and consistent in the true solution $u = \mathcal{L}^{-1}\binom{f}{g}$ of the problem $Lu = f$ and $Bu = g$. Then, the scheme converges in the sense*

$$\|\mathbf{e}_h\|_{h,\infty} \xrightarrow[h\to 0]{} 0,$$

and the order of consistency is equal to the order of convergence.

The definitions and results given above are rather general, i.e. they may be directly generalized to multi–dimensional problems. For the one–dimensional case, we have

Theorem 2.25. *The difference scheme defined by $\mathcal{L}_h$ for the 1-D problem is consistent in every $u \in C^{2+k,\alpha}(\bar{\Omega})$ with $m \in \mathbb{N}_0$ and $\alpha > 0$. The order of consistency is* $\min\{2, k+\alpha\}$.

Proof by Taylor expansion (see Exercise 2.8). ∎

We observe, that strong regularity results on the true solution are needed in order to obtain a high order of consistency for the finite difference scheme.

In the next step, we try to find necessary conditions for the stability of the finite difference scheme. In particular, we have to estimate the norm of $\mathcal{L}_h^{-1}$. We know already, that $\mathcal{L}_h$ approximates $\mathcal{L}$ for $h \to 0$ so that we can assume that the inverse operator $\mathcal{L}_h^{-1}$ also behaves similar to $\mathcal{L}^{-1}$ in that limit. Thus, it is not surprising, that we need detailed knowledge about the exact solution operator to draw conclusions about the discretized one. In Section 2.2, we have seen that $\mathcal{L}^{-1}$ satisfies a bound exactly like the one we ask of $\mathcal{L}_h^{-1}$. In particular, a suitable norm is the maximum norm which is the reason for using that norm also in the discrete case. The derivation of the bound itself was based on the maximum principle which again is just a result of the positivity of the solution operator $\mathcal{L}^{-1}$. Therefore, it is natural, to investigate the positivity properties of the discretization. It turns out, that this investigation yields the desired invertibility and the bound. Since we need a few aspects from the theory of positive operators, we slightly divert from the stability problem and consider *positive matrices* (which are the finite dimensional counterpart of positive operators).

Definition 2.26. *On $\mathbb{R}^N$, we define an order relation $\geq$ by*

$$\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \geq \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \qquad \Longleftrightarrow \qquad y_i \geq x_i \quad i = 1, \ldots, N.$$

If $y \geq x$ and $y \neq x$, the usual notation is $y > x$. In spaces of dimension bigger than one, there is a third notion $y \gg x$ which describes the case $y_i > x_i$ for all $i = 1, \ldots, N$. Finally, a matrix $B \in \mathbb{R}^{N \times N}$ is called positive ($B \geq 0$), if

$$Bx \geq 0 \qquad \forall x \geq 0.$$

Geometrically, the set of all positive vectors forms a cone in $\mathbb{R}^N$ (in $\mathbb{R}^2$ it is the first quadrant). While strictly positive vectors $x > 0$ can still be located on the boundary of that cone, strongly positive vectors $x \gg 0$ point into the interior. By taking the unit vectors e_i, we immediately find that positive matrices are exactly those matrices which have nonnegative entries. We further have

Lemma 2.27. *Positive multiples, sums and products of positive matrices are again positive. Positive matrices are monotone in the sense that if $B \geq 0$ then*

$$y \geq x \Longrightarrow By \geq Bx.$$

Proof: Exercise 2.9. ■

Similar to the scalar case, one defines order intervals $[a, b] \subset \mathbb{R}^N$ for $a, b \in \mathbb{R}^N$ by

$$x \in [a, b] \qquad \Longleftrightarrow \qquad a \leq x \leq b.$$

In particular, if $y \gg 0$, the order interval $[-y, y]$ can be considered as the unit ball of some norm $\| \cdot \|_y$ on $\mathbb{R}^N$. If we take for example the typical strictly positive vector $e = (1, \ldots, 1)^T$, then $\| \cdot \|_e$ is just the maximum norm since $x \in [-e, e]$ implies for all i that $-1 \leq x_i \leq 1$ so that $\|x\|_\infty \leq 1$ which means that x is in the $\| \cdot \|_\infty$ unit ball. For more general $y \gg 0$ we have

Lemma 2.28. *Let $y \gg 0$ and define*

$$\|x\|_y := \max_{i=1}^{N} \frac{|x_i|}{y_i}, \qquad x \in \mathbb{R}^N.$$

Then, the order interval $[-y, y]$ is the unit ball with respect to the $\|\cdot\|_y$. Defining the matrix norm

$$\|B\|_y = \max_{\|x\|_y = 1} \|Bx\|_y$$

we have for any positive matrix $B \geq 0$

$$\|B\|_y = \|By\|_y.$$

In particular, for $y = e = (1, \ldots, 1)^T$, we find

$$\|Bx\|_\infty \leq \|Be\|_\infty \|x\|_\infty \qquad \forall x \in \mathbb{R}^N.$$

Proof: It is easy to check that $\| \cdot \|_y$ defines a norm on $\mathbb{R}^N$. Now, let $-y \leq x \leq y$, i.e. $|x_i| \leq y_i$ for all i. Then, since $y \gg 0$, we have $\frac{|x_i|}{y_i} \leq 1$ and thus $\|x\|_y \leq 1$. Conversely, if $\|x\|_y \leq 1$, we conclude that $-y \leq x \leq y$ so that

$$[-y, y] = \{x \, : \, \|x\|_y \leq 1\}.$$

If $B \geq 0$, then $-y \leq x \leq y$ implies $-By \leq Bx \leq By$ and thus also

$$\frac{|(Bx)_i|}{y_i} \leq \frac{(By)_i}{y_i}$$

so that $\|Bx\|_y \leq \|By\|_y$. Using this relation for those $x \in [-y, y]$ which satisfy $\|x\|_y = 1$, we get by definition of the matrix norm $\|B\|_y \leq \|By\|_y$. Since also $\|y\|_y = 1$, we even have $\|B\|_y = \|By\|_y$. ■

An immediate consequence of this Lemma is that positive matrices B which are contracting in the sense $By \ll y$ are also contracting in the $\| \cdot \|_y$–norm because $B[-y, y]$ is strictly contained in $[-y, y]$, or in other words, $\|B\|_y = \|By\|_y < \|y\|_y = 1$. We thus get

Lemma 2.29. *Let $y \gg 0$. If $B \geq 0$ with $By \ll y$, then $(I - B)^{-1}$ exists and is positive. Similarly, if $By \gg 0$ and $(I - B) \geq 0$ then B^{-1} exists and is positive.*

Proof: Since $\|B\|_y < 1$, the von Neumann series

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k$$

converges in the $\| \cdot \|_y$ norm (and thus in any matrix norm). Together with B, each power is positive so that the same holds for the series. If $By \gg 0$, we can write

$$(I - B)y = y - By \ll y$$

so that the result follows if we replace B by $(I - B)$ in the argument above. ■

With this result, we have established the connection between positivity properties and invertibility of a matrix. Considering the matrix $\mathcal{L}_h$, we first note that the diagonal terms are always strictly positive. Indeed, the top and bottom entry is equal to one and the other entries are of the form $d_i = 2/h^2 + c_i$ which is certainly positive since $c_i = c(x_i) \geq 0$ and $h > 0$. The off-diagonal terms, on the other hand, are non positive, provided

$$h \leq \frac{2}{\max_{\bar{\Omega}} |b|} =: h_0. \tag{2.16}$$

(A matrix whose off-diagonal elements are non positive is sometimes called an L_0–matrix.) If we collect the diagonal elements of $\mathcal{L}_h$ in the matrix $\mathcal{D}$, then $\mathcal{D}$

is invertible due to the strict positivity. Of course, we also have $\mathcal{D}, \mathcal{D}^{-1} \geq 0$. By subtracting $\mathcal{D}$ from $\mathcal{L}_h$, we isolate the non positive terms, i.e.

$$\mathcal{D} - \mathcal{L}_h \geq 0.$$

Multiplying this inequality by $\mathcal{D}^{-1} \geq 0$, we get

$$I - \mathcal{D}^{-1}\mathcal{L}_h \geq 0.$$

To apply Lemma 2.29, we just need a vector $y \gg 0$ such that $\mathcal{L}_h y \gg 0$ (since then also $\mathcal{D}^{-1}\mathcal{L}_h y \gg 0$). Given such a vector y, we can conclude that $\mathcal{D}^{-1}\mathcal{L}_h$ has a positive inverse so that also $\mathcal{L}_h^{-1}$ exists and is positive. To estimate $\mathcal{L}_h^{-1}$ is the maximum norm, we observe that

$$\mathcal{L}_h y \geq \min_{i=1}^{N} (\mathcal{L}_h y)_i e, \qquad e = (1, \ldots, 1)^T.$$

Multiplying with $\mathcal{L}_h^{-1}$ yields

$$0 \leq \mathcal{L}_h^{-1} e \leq \frac{y}{\min_{i=1}^{N} (\mathcal{L}_h y)_i}$$

which finally gives

$$\|\mathcal{L}_h^{-1}\|_{h,\infty} = \|\mathcal{L}_h^{-1} e\|_{h,\infty} \leq \frac{\|y\|_{h,\infty}}{\min_{i=1}^{N} (\mathcal{L}_h y)_i}. \tag{2.17}$$

The remaining problem is the construction of a suitable vector $y \gg 0$. Of course, y has to be chosen in such a way that its components stay bounded for $h \to 0$ (to bound $\|y\|_{h,\infty}$ in (2.17)). In addition, $\mathcal{L}_h y$ must be bounded away from zero. The latter condition could be obtained by taking for y the solution of

$$\mathcal{L}_h y = e, \qquad e = (1, \ldots, 1)^T.$$

While this would guarantee the required lower bounds, we are not sure about the norm of the solution y. The trick is to resort to the corresponding continuous problem which we have much better under control. Since the data vector $d_h = e$ is a discretization of $d = \binom{1}{1}$ (i.e. $f \equiv 1$ and $g \equiv 1$), the continuous problem is

$$\begin{aligned} Lv &= 1 \qquad \text{in } \Omega \\ v &= 1 \qquad \text{on } \partial\Omega \end{aligned}$$

Applying the minimum principle (since $Lv \geq 0$), we get $v \geq 1$ and since v is continuous, we also find $\|v\|_\infty = \sup_{\bar{\Omega}} |v| < \infty$. Setting

$$y = D_h v,$$

we get $y \gg e \gg 0$ (independent of h, i.e. the dimension of the discrete space), and also

$$\mathcal{L}_h y = D_h \mathcal{L} v + \mathbf{l}_h(v) = e + \mathbf{l}_h(v) \gg \frac{1}{2} e, \qquad \text{for } h \text{ small enough.}$$

(Here, we need that $\|\mathbf{l}_h(v)\|_{h,\infty} \to 0$ for $h \to 0$ for the solution v of the auxiliary problem.) Using the so defined vector y in connection with (2.17), we end up with the bound

$$\|\mathcal{L}_h^{-1}\|_{h,\infty} \leq 2\|v\|_\infty$$

which proves that the scheme is stable. We summarize the result in

Theorem 2.30. *Under the general smoothness assumptions, the finite difference scheme defined by $\mathcal{L}_h$ is convergent. If the solution is contained in $C^{2+k,\alpha}(\bar{\Omega})$, the order of convergence is $\min\{2, k+\alpha\}$.*

We remark that extra smoothness, which leads to higher order of convergence, is typically connected to a corresponding regularity of the data, the coefficients of L and the domain Ω (see Section 2.5).

Before we go over to 2D problems, let us first investigate the matrix $\mathcal{L}_h$ a little closer. To get an idea about the structure of $\mathcal{L}_h$, we consider its eigenvalues (which essentially describe the action of the matrix). To this end, the following result is very useful.

Lemma 2.31. (Gerschgorin)
Let $A \in \mathbb{C}^{N\times N}$ be a complex matrix. Then, the eigenvalues of A are contained in the union of closed disks $\overline{B_{r_i}(A_{ii})}$ in the complex plane, where $r_i = \sum_{j\neq i} |A_{ij}|$.

Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with a corresponding eigenvector x which has been normalized to one in the norm

$$\|x\|_\infty := \max_{i=1}^{N} |x_i|.$$

Hence, there is at least one index i, such that $|x_i| = 1$ and thus

$$(\lambda - A_{ii})x_i = \sum_{j\neq i} A_{ij}x_j.$$

Taking absolute values on both sides and using $|x_i| = 1$, $|x_j| \leq 1$ for $i \neq j$ yields

$$|\lambda - A_{ii}| \leq \sum_{j\neq i} |A_{ij}| = r_i.$$

■

For the matrix $\mathcal{L}_h$, we find

$$\sigma(\mathcal{L}_h) \subset \bigcup_{i=1}^{n-1} B_{|l_i|+|r_i|}(d_i) \cup \{1\}.$$

For $h < h_0$ (defined in (2.16)), we get

$$\left|\frac{b_i}{2}\right| < \frac{1}{h}$$

so that

$$|l_i| = \frac{1}{h^2} + \frac{b_i}{2h}, \qquad |r_i| = \frac{1}{h^2} - \frac{b_i}{2h}$$

and hence $|l_i| + |r_i| = 2/h^2$. Consequently,

$$\sigma(\mathcal{L}_h) \subset \bigcup_{i=1}^{n-1} B_{2/h^2}\left(\frac{2}{h^2} + c_i\right) \cup \{1\},$$

i.e. all eigenvalues are contained in the positive half space $\{z \in \mathbb{C} : \Re(z) \geq 0\}$. While $c_i > 0$ for all i implies invertibility of $\mathcal{L}_h$ (since $0 \notin \sigma(\mathcal{L}_h)$), the case $c_i = 0$ for some i needs a more involved analysis than the Lemma of Gerschgorin. (We proved it based on a contraction property.) The case $c_i < 0$, on the other hand, can definitely lead to $0 \in \sigma(\mathcal{L}_h)$ for special choices of c.

We conclude the section on 1D problems with the remark that instead of $\mathcal{L}_h$ we can also derive another linear system which just operates on the interior points $x_1, \ldots, x_{n-1}$. In some sense, it is therefore a discretization of the continuous operator L. However, the boundary values enter in the right hand side, which has no direct counterpart in the original problem. To derive that system, we use the fact that u_0 and u_n are already known to be g_0 and g_n. Extracting the system for the remaining components $u_1, \ldots, u_{n-1}$, we find the equivalent set of equations

$$\begin{pmatrix} d_1 & r_1 & & \\ & \ddots & \ddots & \ddots \\ & & l_{n-1} & d_{n-1} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} f_1 - l_1 g_0 \\ \vdots \\ f_{n-1} - r_{n-1} g_n \end{pmatrix}$$

which we abbreviate $\bar{\mathcal{L}}_h \bar{\mathbf{u}}_h = \bar{\mathbf{d}}_h$. In contrast to the original matrix, $\bar{\mathcal{L}}_h$ can be symmetric if $r_i = l_{i+1}$, i.e.

$$-\frac{1}{h^2} + \frac{b_i}{2h} = -\frac{1}{h^2} - \frac{b_{i+1}}{2h} \quad \Longleftrightarrow \quad b_i = -b_{i+1}.$$

In particular, if $b = 0$, the latter condition is satisfied. For the important case of the Laplacian, we have in addition $c = 0$ so that

$$\bar{\mathcal{L}}_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{pmatrix}$$

Since $\bar{\mathcal{L}}_h$ is symmetric, all eigenvalues are real and due to the Lemma of Gerschgorin, they are not negative. Finally, the invertibility of $\bar{\mathcal{L}}_h$ excludes the eigenvalue zero so that $\bar{\mathcal{L}}_h$ is positive definite. In order to solve the approximate system it is therefore advisable to use the form $\bar{\mathcal{L}}_h$ because very efficient numerical algorithms exist for positive definite symmetric matrices.