2.8. Fundamental solutions and Green's functions

Definition 2.37. *Let $L = \sum_{|\alpha|\le N} c_\alpha \nabla^\alpha$ be a differential operator with constant coefficients on $\mathbb{R}^n$. Then a fundamental solution for L is a distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ satisfying the equation $LE = \delta$.*

Of course, fundamental solutions are unique only up to a solution of the homogeneous equation $Lu = 0$. Typically, one chooses a particular solution with a behavior at infinity which is as nice as possible. The significance of the fundamental solution lies in the fact that it can be used to solve general problems of the form

$$Lu = f, \qquad f \in \mathscr{D}'(\mathbb{R}^n), \quad \text{supp } f \text{ compact.}$$

Indeed, setting $u = E * f$, we find

$$Lu = L(E * f) = (LE) * f = \delta * f = f. \tag{2.27}$$

The *Theorem of Malgrange and Ehrenpreis* states that any differential operator with constant coefficients admits a fundamental solution [**15**]. Here, we show that for the case $L = \Delta$, a fundamental solution is given by

$$N(x) = g(|x|) = \begin{cases} \frac{1}{\omega_1}|x| & n = 1 \\ \frac{1}{\omega_2}\ln|x| & n = 2 \\ \frac{1}{(2-n)\omega_n}|x|^{2-n} & n > 2. \end{cases}$$

(For other examples we refer to [**13**].) Indeed, since $N \in \mathbb{L}^1_{loc}(\mathbb{R}^n)$, we have for any test function ϕ

$$\langle \Delta N, \phi\rangle = \langle N, \Delta\phi\rangle = \int_{\mathbb{R}^n} N(x)\Delta\phi(x)\,dx.$$

Using Lemma 1.4 with $\Omega = B_R(0)$ bigger than the support of ϕ (so that $\phi = 0$ and $\partial\phi/\partial n = 0$ on $\partial\Omega$), we get

$$\int_{\mathbb{R}^n} N(x)\Delta\phi(x)\,dx = \phi(0) = \langle \delta, \phi\rangle$$

which shows that $\Delta N = \delta$. Using the important relation (2.27), we recover the result of Section 1.3.2 that $\Phi = N * \rho$ is a solution of the Poisson problem $\Delta\Phi = \rho$ on $\mathbb{R}^n$. If ρ is a compactly supported, generalized function on $\mathbb{R}^n$, the equation is satisfied in the sense of distributions. If ρ has more regularity (like $C_0^2(\mathbb{R}^n)$), the equation is even satisfied in the classical sense (see Theorem 1.11).

The adjective *fundamental* for the solution N of $\Delta N = \delta$ is justified in the sense that it can be used as building block for solutions of more general equations. In particular, (2.27) shows that a solution operator of the differential equation $\Delta u = f$ is given by the integral operator $u = N * f$. (Note that convolution of two distributions can, in general, not be written as an integral. It only generalizes an integral expression.) Apart from the full space problem, the convolution $N * f$ also yields solutions to problems on bounded domains.

Assume, for example, that $f \in \mathscr{D}'(\Omega)$ is the restriction of a compactly supported distribution $\hat{f}$ on $\mathbb{R}^n$ to the set Ω. (If Ω is bounded, any $f \in \mathbb{L}^\infty(\Omega)$ falls in this class since the extension by zero of f to $\mathbb{R}^n$ is a compactly supported generalized function on $\mathbb{R}^n$.) Using the previous result, we get again that

$$v = N * \hat{f}\Big|_\Omega$$

satisfies $\Delta v = f$ in $\mathscr{D}'(\Omega)$. In order to satisfy also boundary conditions on $\partial\Omega$, however, this approach has to be modified. First of all, we have to require enough regularity on f in order to refer to values of $N * \hat{f}$ on the boundary. Here, we take $f \in \mathbb{L}^\infty(\Omega)$ and Ω bounded. Then, the convolution can be written as

$$v(x) = N * \hat{f}(x) = \int_\Omega N(x-y) f(y)\, dy, \qquad x \in \Omega.$$

(The restricted domain of integration is due to the fact that we convolute N with $\hat{f}$ which is equal to f on Ω and zero in Ω^c.) If we want to solve the Dirichlet problem

$$\begin{aligned} \Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2.28}$$

we obviously have to modify v (which already satisfies $\Delta v = f$) by adding a harmonic function h with the property $h = -v$ on $\partial\Omega$. More generally, if $H(x,y)$ is harmonic in x for each $y \in \Omega$ and satisfies $H(x,y) = -N(x-y)$ whenever $x \in \partial\Omega$ and $y \in \Omega$, then

$$u(x) = \int_\Omega \left(N(x-y) + H(x,y)\right) f(y)\, dy$$

satisfies (2.28). This idea gives rise to the following

Definition 2.38. *Let Ω be an open subset of $\mathbb{R}^n$. A function $G : \bar{\Omega} \times \Omega \mapsto \mathbb{R}$ is called Green's function, if*

1) *$x \mapsto H(x,y) = G(x,y) - N(x-y)$ is harmonic in Ω and continuous in $\bar{\Omega}$ for all $y \in \Omega$,*
2) *$G(x,y) = 0$ for all $x \in \partial\Omega$ and $y \in \Omega$.*

Note that, on bounded domains, the Green's function is uniquely defined since $x \mapsto H(x,y)$ is the classical solution of a Dirichlet Laplace problem which is unique. Since the singularity of G is exactly the one of the fundamental solution, the following result is not surprising.

Lemma 2.39. *Let G be a Green's function on $\Omega \subset \mathbb{R}^n$. Then*

$$\Delta_x G(x,y) = \delta(x-y), \qquad \forall y \in \Omega$$

holds in $\mathscr{D}'(\Omega)$.

Proof: For given $y \in \Omega$, we can find $\epsilon > 0$ such that $B_{3\epsilon}(y) \subset \Omega$. Thus, we can construct a function $\eta \in \mathscr{D}(\Omega)$ which is equal to one on $B_\epsilon(y)$ and equal to zero outside of $B_{2\epsilon}(y)$. Picking an arbitrary test function $\phi \in \mathscr{D}(\Omega)$, we have

$$\langle \Delta_x G(x,y), \phi(x) \rangle_x = \langle \Delta_x G(x,y), \eta(x)\phi(x) \rangle_x + \langle \Delta_x G(x,y), (1-\eta(x))\phi(x) \rangle_x .$$

Using the fact that $\Delta_x G(x,y) = 0$ for x away from y, we find

$$\langle \Delta_x G(x,y), \phi(x) \rangle_x = \langle \Delta_x G(x,y), \eta(x)\phi(x) \rangle_x .$$

The splitting $G(x,y) = N(x-y) + H(x,y)$ then yields (due to $\Delta_x H(x,y) = 0$)

$$\langle \Delta_x G(x,y), \phi(x) \rangle_x = \langle \Delta_x N(x-y), \eta(x)\phi(x) \rangle_x .$$

Since N is a fundamental solution and $\eta(y) = 1$, we end up with

$$\langle \Delta_x G(x,y), \phi(x) \rangle_x = \langle \delta(x-y), \eta(x)\phi(x) \rangle_x = \langle \delta(x-y), \phi(x) \rangle_x .$$

■

We remark that the equation

$$L_x G(x,y) = \delta(x-y), \qquad \forall y \in \Omega \tag{2.29}$$

is also meaningful for general differential operators (with C^∞ coefficients), giving rise to Green's functions for L. However, the question of suitable boundary conditions (and whether solutions to (2.29) have enough regularity to talk about boundary values) is quite complicated in the general case. For a formal discussion we refer to [**14**]. Due to Definition 2.38, which gives additional structure of G for the special case of the Laplacian, we can, however, infer several results.

Lemma 2.40. *Let Ω be bounded, $f \in \mathbb{L}^\infty(\Omega)$ and G be the Green's function. Then*

$$u(x) = \int_\Omega G(x,y) f(y)\, dy, \qquad x \in \Omega$$

defines a function in $C(\bar{\Omega})$ which solves $\Delta u = f$ in $\mathscr{D}'(\Omega)$ and satisfies $u = 0$ on $\partial\Omega$. If $x \mapsto H(x,y)$ is contained in $C^1(\bar{\Omega})$, then also $u \in C^1(\bar{\Omega})$ and

$$\nabla u(x) = \int_\Omega \nabla_x G(x,y) f(y)\, dy, \qquad x \in \Omega.$$

Proof: The regularity of u is obtained from Lemma 2.14 which shows that integrals over the fundamental solution yield $C^1(\bar{\Omega})$ functions. If H has also sufficient regularity, the same holds for u. Taking any test function $\phi \in \mathscr{D}(\Omega)$, we get

$$\langle \Delta u, \phi \rangle = \langle u, \Delta\phi \rangle = \int_\Omega \int_\Omega G(x,y) f(y) \Delta\phi(x)\, dy\, dx$$
$$= \int_\Omega \langle G(x,y), \Delta\phi(x) \rangle_x f(y)\, dy.$$

Using Lemma 2.39, we have

$$\langle \Delta u, \phi \rangle = \int_{\Omega} \phi(y) f(y)\, dy = \langle f, \phi \rangle$$

which shows that $\Delta u = f$ in $\mathscr{D}'(\Omega)$. The Dirichlet condition follows directly from property 2) in Definition 2.38. ■

Before we consider the case of more general Dirichlet conditions, let us first investigate the smoothness of the solution u in Lemma 2.40 if the source f has extra smoothness. It turns out that the Laplacian Δ is a *hypoelliptic* operator which means that $\Delta u \in C^\infty$ implies $u \in C^\infty$. In general, we have

Definition 2.41. *The linear partial differential operator L in Ω is said to be hypoelliptic if for any open subset $U \subset \Omega$ and $u \in \mathscr{D}'(\Omega)$, $Lu \in C^\infty(U)$ implies $u \in C^\infty(U)$.*

Suppose that E is any fundamental solution of L in $\mathbb{R}^n$. Consequently, $LE = 0$ in $\mathbb{R}^n \backslash \{0\}$. Hence, if L is hypoelliptic, E must be a C^∞ function in $\mathbb{R}^n \backslash \{0\}$. A classical theorem of L. Schwartz states that the converse is also true.

Theorem 2.42. *If there exists a fundamental solution E of L which is a C^∞–function in $\mathbb{R}^n \backslash \{0\}$ then L is hypoelliptic in $\mathbb{R}^n$.*

Proof: Let $U \subset \mathbb{R}^n$ be an open set and assume $u \in \mathscr{D}'(U)$ satisfies $Lu = f \in C^\infty(U)$. Picking any $x_0 \in U$, we can find $\epsilon > 0$ so that $B_{5\epsilon}(x_0) \subset U$. Selecting a function $\eta \in \mathscr{D}(\mathbb{R}^n)$ which is equal to one on $B_{3\epsilon}(x_0)$ and zero outside of $B_{4\epsilon}(x_0)$, we restrict the distribution u to a neighborhood of x_0 by taking the product ηu. Applying the differential operator, we get

$$L(\eta u) = \eta L u + v = \eta f + v$$

where v is obtained with Leibniz' formula. In particular, v vanishes on $B_{2\epsilon}(x_0)$ and outside of $B_{4\epsilon}(x_0)$ since the derivatives of η are zero in these sets (v is a sum of terms $\nabla^\alpha \eta \nabla^\beta u$ with $|\alpha| \geq 1$). Taking another cut–off function χ which is one on $B_\epsilon(0)$ and vanishes outside $B_{2\epsilon}(0)$, we finally get

$$\eta u = \delta * (\eta u) = (LE) * (\eta u) = E * (L(\eta u)) = E * (\eta f) + E * v$$

and, by introducing χ,

$$\eta u = E * (\eta f) + (\chi E) * v + ((1 - \chi)E) * v.$$

The first term on the right hand side is a convolution of a test function ηf and a distribution E, i.e. $E * (\eta f) \in C^\infty(\mathbb{R}^n)$. The third term is a convolution of a compactly supported distribution v and a C^∞–function $(1-\chi)E$ and thus also contained in $C^\infty(\mathbb{R}^n)$. Finally, the middle term is a distribution whose support is contained in

$$\operatorname{supp}(\chi E) * v \subset \operatorname{supp}(\chi E) + \operatorname{supp} v \subset B_\epsilon(0) + B_{2\epsilon}(x_0)^c \subset B_\epsilon(x_0)^c.$$

In particular, $(\chi E) * v$ vanishes on $B_\epsilon(x_0)$ and thus ηf is a C^∞–function on this neighborhood. Since $\eta \equiv 1$ on $B_\epsilon(x_0)$ and x_0 was an arbitrary point in U, it follows $u \in C^\infty(U)$ and the proof is complete. ∎

The result applied to $L = \Delta$ is known as Weyl's Lemma.

Corollary 2.43. **(Weyl's Lemma)**
The Laplace operator is hypoelliptic on $\mathbb{R}^n$.

Proof: N is a C^∞ function on $\mathbb{R}^n \backslash \{0\}$. ∎

Note that this corollary yields regularity results on $\mathscr{D}'(\Omega)$ solutions to the Poisson problem. If Ω is an open subset of $\mathbb{R}^n$, $f \in C^\infty(\Omega)$ and $\Delta u = f$ in the sense of distributions then, automatically, $u \in C^\infty(\Omega)$. Since N is even analytic in $\mathbb{R}^n \backslash \{0\}$, one can show that Δ is *analytic –hypoelliptic*. The definition is similar to the one of hypoellipticity with C^∞ replaced by analyticity (see [**17**]).

Before we continue with the solution formula for general Dirichlet Laplace problems $\Delta u = 0$ in Ω and $u = g$ on $\partial\Omega$ we need some symmetry result on G.

Lemma 2.44. *Let $\Omega \subset \mathbb{R}^n$ be open and let G be the Green's function on Ω. Then, for any pair of points $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$, we find*

$$G(x_1, x_2) = G(x_2, x_1).$$

Proof: Since x_1 is different from x_2, we can find $\epsilon > 0$ such that $B_{4\epsilon}(x_2) \subset \Omega$ and $x_1 \notin B_{4\epsilon}(x_2)$. We then construct a function $\phi \in \mathscr{D}(\Omega)$ which is equal to one on $B_{2\epsilon}(x_2)$ and $\phi = 0$ outside of $B_{3\epsilon}(x_2)$ so that

$$\begin{aligned} G(x_2, x_1) &= \phi(x_2) G(x_2, x_1) = \langle \delta(x - x_2), \phi(x) G(x, x_1) \rangle \\ &= \langle \Delta G(x, x_2), \phi(x) G(x, x_1) \rangle = \langle G(x, x_2), \Delta(\phi(x) G(x, x_1)) \rangle . \end{aligned}$$

By construction, $x \mapsto G(x, x_1)$ is a C^∞–function on the support of ϕ. In particular, $x \mapsto G(x, x_1)$ is harmonic on that set. Thus

$$\begin{aligned} &\Delta(\phi(x) G(x, x_1)) \\ &\quad = \phi(x) \Delta G(x, x_1) + \big(G(x, x_1) \Delta \phi(x) + 2 \nabla \phi(x) \cdot \nabla G(x, x_1) \big) \\ &\quad = G(x, x_1) \Delta \phi(x) + 2 \nabla \phi(x) \cdot \nabla G(x, x_1). \end{aligned}$$

Since this expression vanishes in $B_{2\epsilon}(x_2)$ (where $\phi \equiv 1$ so that the derivatives vanish), we can multiply it by a function $\psi \in \mathscr{D}(\Omega)$ which vanishes on $B_\epsilon(x_2)$ and is equal to one outside of $B_{2\epsilon}(x_2)$ in a set Ω_δ which contains x_1 and $B_{4\epsilon}(x_2)$. Thus

$$G(x_2, x_1) = \langle \psi(x) G(x, x_2), \Delta(\phi(x) G(x, x_1)) \rangle$$

and since both expressions are test functions, we can invert the order

$$\begin{aligned} G(x_2, x_1) &= \langle \Delta(\phi(x) G(x, x_1)), \psi(x) G(x, x_2) \rangle \\ &= \langle \phi(x) G(x, x_1), \Delta(\psi(x) G(x, x_2)) \rangle . \end{aligned}$$

If we now take a sequence $(\phi_n)_{n\in\mathbb{N}}$ with $\phi_n \in \mathscr{D}(\Omega\backslash\{x_1\}$ and $0 \le \phi \le \phi_n \le 1$ which converges point wise to one on Ω, we get with Lebesgue's dominated convergence

$$G(x_2,x_1) = \langle \phi_n(x)G(x,x_1), \Delta(\psi(x)G(x,x_2))\rangle \xrightarrow[n\to\infty]{} \langle G(x,x_1), \Delta(\psi(x)G(x,x_2))\rangle .$$

Using the property of the Green's function derived in Lemma 2.39, we obtain

$$G(x_2,x_1) = \langle \delta(x-x_1), \psi(x)G(x,x_2)\rangle = \psi(x_1)G(x_1,x_2) = G(x_1,x_2).$$

■

The following Lemma yields a second application of Green's functions.

Lemma 2.45. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a piece-wise C^1 boundary. Let G be the Green's function on Ω which satisfies*

$$x \mapsto G(x,y) - N(x-y) \in C^1(\bar\Omega) \qquad \forall y \in \Omega.$$

Then, any harmonic function $u \in C^2(\Omega)\cap C^1(\bar\Omega)$ satisfies

$$u(x) = \int_{\partial\Omega} u(y)\frac{\partial}{\partial n_y}G(x,y)\,d\sigma(y), \qquad x\in\Omega.$$

Proof: Since u is harmonic in Ω, we have

$$\int_\Omega G(x,y)\Delta u(y)\,dy = 0. \tag{2.30}$$

Multiplying (2.30) by a test function $\phi \in \mathscr{D}(\Omega)$ and integrating yields after changing the order of integration

$$0 = \int_\Omega \Delta u(y) \int_\Omega G(x,y)\phi(x)\,dx\,dy.$$

Since $G(x,y) = G(y,x)$, the inner integral

$$v(y) = \int_\Omega G(x,y)\phi(x)\,dx$$

is, according to Lemma 2.40, the solution of the problem

$$\begin{aligned} \Delta v &= \phi \qquad \text{in } \Omega \\ v &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2.31}$$

Moreover, $v \in C^1(\bar\Omega)\cap C^\infty(\Omega)$ so that Green's formula can be applied. We find

$$0 = \int_\Omega v\Delta u\,dy = \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,d\sigma(y) - \int_{\partial\Omega} u\frac{\partial v}{\partial n}\,d\sigma(y) + \int_\Omega u\Delta v\,dy.$$

Due to (2.31), we get

$$0 = -\int_{\partial\Omega} un(y)\cdot\nabla_y \int_\Omega G(x,y)\phi(x)\,dx\,d\sigma(y) + \int_\Omega u\phi\,dy.$$

Due to Lemma 2.40, the derivative ∂_y can be taken under the integral so that

$$0 = \int_\Omega \big(u(x) - \int_{\partial\Omega} u(y) \frac{\partial}{\partial n_y} G(y,x)\, d\sigma(y)\big)\phi(x)\, dx.$$

Using again the symmetry $G(x,y) = G(y,x)$, we obtain

$$u(x) = \int_{\partial\Omega} u(y) \frac{\partial}{\partial n_y} G(x,y)\, d\sigma(y), \qquad x \in \Omega.$$

■

This result shows that the solution to general Dirichlet Laplace problems can be obtained through an integral operator on $\partial\Omega$ with kernel $\frac{\partial}{\partial n_y} G(x,y)$, the so called *Poisson kernel.*

Lemma 2.46. *Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$ with $\alpha > 0$. Further let $g \in C(\partial\Omega)$. If G is the Green's function on Ω, then*

$$u(x) = \int_{\partial\Omega} g(y) \frac{\partial}{\partial n_y} G(x,y)\, d\sigma(y), \qquad x \in \Omega$$

is the solution of

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

Proof: We first note that, for the regular domain under consideration, the Green's function has the smoothness properties required in Lemma 2.45. Indeed, $x \mapsto H(x,y) = G(x,y) - N(x-y)$ is a smooth function on $\partial\Omega$ so that with Theorem 2.18, $x \mapsto H(x,y)$ is contained in $C^{2,\alpha}(\bar\Omega)$. For the same reason, $g \in C^{2,\alpha}(\partial\Omega)$ gives rise to a unique solution u of the Dirichlet Laplace problem which is in $C^{2,\alpha}(\bar\Omega)$. Applying the previous Lemma we find the representation formula

$$u(x) = \int_{\partial\Omega} g(y) \frac{\partial}{\partial n_y} G(x,y)\, d\sigma(y), \qquad x \in \Omega$$

If g is only continuous, we go over to an approximating sequence $(g_n)_{n\in\mathbb{N}}$ of smooth functions which converges uniformly to g on $\partial\Omega$. Using the continuous dependence on the data (Theorem 2.8) we obtain convergence of the corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$ to the solution of the limit problem. Applying Lebesgue's dominated convergence yields

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= \lim_{n\to\infty} \int_{\partial\Omega} g_n(y) \frac{\partial}{\partial n_y} G(x,y)\, d\sigma(y) = \int_{\partial\Omega} g(y) \frac{\partial}{\partial n_y} G(x,y)\, d\sigma(y). \end{aligned}$$

■

For domains Ω with a particularly simple geometry, the Green's function can sometimes be guessed. For the half space $\Omega = \{x \in \mathbb{R}^n : x_1 > 0\}$, the Green's function is obtained by the *method of images.* To construct a harmonic function

$x \mapsto H(x,y)$ which has the values $-N(x-y)$ for x on the boundary $\partial\Omega$ we evaluate the fundamental solution at the *reflected* point

$$\hat{x} = (-x_1, x_2, \dots, x_n)^T.$$

More precisely, for any $y \in \Omega$, we set

$$H(x,y) = -N(\hat{x} - y), \qquad x \in \bar{\Omega}.$$

Then, for $x \in \partial\Omega$, i.e. $x = (0, x_2, \dots, x_n)^T$ we have $x = \hat{x}$ so that

$$G(x,y) = N(x-y) + H(x,y) = N(x-y) - N(\hat{x} - y) = 0$$

which is the second condition in Definition 2.38. The other requirements follow easily from the fact that $|\hat{x} - y| = |x - \hat{y}|$ which implies that $x \mapsto H(x,y) = N(x - \hat{y})$ is harmonic in $\bar{\Omega} \subset \mathbb{R}^n \setminus \{\hat{y}\}$. For the example of a ball in $\mathbb{R}^n$ and the quarter plane in 2D see Exercises 2.18 and 2.19. We remark that the Poisson kernel of the Green's function on the ball has already been investigated in Theorem 1.14.

Instead of the Dirichlet problem one can also consider other boundary conditions with Green's functions. For the Laplace operator and Neumann conditions only 2) in Definition 2.38 has to be changed to $\frac{\partial}{\partial n_x} G(x,y) = 0$ for $x \in \partial\Omega$ and $y \in \Omega$ (plus some extra smoothness on $G(x,y) - N(x-y)$). The construction in particular cases is done similar to the reflection principle above by placing a reflected fundamental solution outside the domain without changing the sign.

We conclude our considerations with the remark that Green's functions can be helpful in special situations where they are easily constructed. Otherwise, the construction of G involves the solution of infinitely many Dirichlet (or Neumann) Laplace problems (for each $y \in \Omega$). In these general cases, showing existence of the Green's function (including smoothness requirements) is as complicated as showing these properties directly for specific Dirichlet (or Neumann) Poisson problems. Thus, Green's functions are, in general, not a means to show existence or regularity of solutions to Dirichlet or Neumann problems. They nicely show, however, that the solution operators to general Dirichlet or Neumann Poisson problems are, in fact, integral operators with weakly singular kernels.

2.9. Exercises

Exercise 2.1. A function $u \in C^0(\bar{\Omega})$ is called *subharmonic*, if for every ball B with $\bar{B} \subset \Omega$ and every function $h \in C(\bar{B})$ with h harmonic in B and $u \leq h$ on ∂B, we have $u \leq h$ in B. Replacing $\leq$ by $\geq$, we also define *superharmonic* functions.

a) Show that this definition generalizes the one from Exercise 1.6.
b) Show that sum and maximum of two subharmonic functions are subharmonic.

c) Show the maximum principle for subharmonic functions: if u is subharmonic with $u \leq 0$ on $\partial\Omega$, then either $u \equiv 0$ or $u < 0$ in Ω, if Ω is connected.

Exercise 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. On its complement, we consider the (exterior) Dirichlet problem

$$Lu = f \quad \text{in } \Omega^c, \qquad\qquad u = g \quad \text{on } \partial\Omega$$

with continuous functions f and g and a general linear, second order operator L whose zeroth order part is non negative. The boundary condition at infinity is

$$|u(x)| \to 0 \qquad \text{for } |x| \to \infty.$$

Show that the solution to this problem is unique if it exists.

Exercise 2.3. Let Ω be open, bounded and connected with a piecewise C^1 boundary. Further, let $f \in C(\bar{\Omega})$ and $g, g_0, g_1, a \in C(\partial\Omega)$ with $a > 0$. Show that the following problems have at most one solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$:

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + au = g & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g_0 & \text{on } \emptyset \neq \Gamma_0 \subset \partial\Omega \\ \dfrac{\partial u}{\partial n} = g_1 & \text{on } \Gamma_1 = \partial\Omega \backslash \Gamma_0 \end{cases}$$

What about the case $\Gamma_0 = \emptyset$?

Exercise 2.4. Show that, in contrast to Δ, the biharmonic operator Δ^2 does not satisfy a maximum principle.

Exercise 2.5. Let u be a classical solution of $\Delta u = u^3 - u$ on a bounded domain Ω. Assume that $u = 0$ on $\partial\Omega$. Show that $u \in [-1, 1]$ throughout Ω. Can the values ± 1 be achieved?

Exercise 2.6. Show that convex sets and sets with a C^2 boundary satisfy the exterior sphere property. A set $\Omega \subset \mathbb{R}^n$ has a C^2–boundary, if the boundary can be locally described by C^2 mappings. More precisely, for any point $x \in \partial\Omega$ there should be some $r > 0$, a rotation matrix R and $\phi \in C^2(B_r(0; \mathbb{R}^{n-1}))$ such that the rotated and translated set $R(\Omega - x)$ can be written as

$$R(\Omega - x) \cap B_r(0) = \{\, (y, z) \in B_r(0) \;:\; y \in \mathbb{R}^{n-1},\ |y| < r,\ z < \phi(y) \,\}.$$

Exercise 2.7. Consider the general linear, second order, elliptic Dirichlet problem

$$\begin{aligned} \tilde{L}u &= \tilde{f} \qquad \text{in } [\alpha, \beta] \\ u &= \tilde{g} \qquad \text{on } \{\alpha, \beta\} \end{aligned}$$

where $\tilde{L} = -au'' + \tilde{b}u' + \tilde{c}u$. Show that under the assumption $a \in C^0([\alpha, \beta])$ with $\min_{[\alpha,\beta]} a > 0$, the problem can be reduced to

$$\begin{aligned} -v'' + bv' + cv &= f \qquad \text{in } [0,1] \\ v &= g \qquad \text{on } \{0,1\} \end{aligned}$$

Exercise 2.8. Show that the difference scheme, defined by $\mathcal{L}_h$ in the lecture, is consistent in every $u \in C^{2+k,\alpha}(\bar{\Omega})$ with $k \in \mathbb{N}_0$ and $\alpha > 0$. The order of consistency is $\min\{2, k+\alpha\}$. ($C^{2+k,\alpha}(\bar{\Omega})$ means that the derivative $2+k$ is Hölder continuous of order α up to the boundary.)

Exercise 2.9.

1) Show that positive multiples, sums and products of positive matrices are again positive.
2) Show that for any positive matrix $B \geq 0$, we have $By \geq Bx$ whenever $y \geq x$.
3) How does the set of positive vectors in $\mathbb{R}^3$ look like?
4) What is the shape of order intervals in $\mathbb{R}^3$?

Exercise 2.10. Show that the assumption $c \geq 0$ can, in general, not be removed from the assumptions for the maximum principle. (Consider the operator $Lu = -u'' - u$ on $(0, \pi)$.)

Exercise 2.11. Let $\Omega = (0, \pi)$, $Lu = -u''$, $Bu = u|_{\partial\Omega}$ and $\mathcal{L} = \binom{L}{B}$.

a) Show that the eigensolutions $u^{(m)}(x) = \sin(mx)$, $m \in \mathbb{N}$ of $\mathcal{L}$ turn into eigenvectors of $\mathcal{L}_h$ under discretization.
b) What is the relation between the discretizations of $u^{(m)}$ and $u^{(m+n)}$ if $h = \pi/n$?
c) Give a list of all eigenvalues of $\mathcal{L}_h$.
d) How do the eigenvalues behave for $h \to 0$?
e) Estimate the condition number $\|\mathcal{L}_h\| \cdot \|\mathcal{L}_h^{-1}\|$ of $\mathcal{L}_h$ for small h.
f) The problem $Lu = u$ on Ω and $u = 0$ on $\partial\Omega$ is not uniquely solvable. What about the corresponding discretized problem?

Exercise 2.12. The matrix $A \in \mathbb{R}^{N \times N}$ is called

1) L_0–matrix, if $a_{ij} \leq 0$ for all $i \neq j$,
2) M–matrix, if A is an L_0–matrix, A^{-1} exists, and A^{-1} is positive.

Show that the L_0–matrix A is an M–matrix if and only if there exists $y \gg 0$ such that $Ay \gg 0$. Moreover

$$\|A^{-1}\|_\infty \leq \frac{\|y\|_\infty}{\min_k (Ay)_k}.$$

Hint: Consider the stability proof for the 1D finite difference scheme.

Exercise 2.13. Construct a fourth order consistent approximation for the second derivative in 1D.

Exercise 2.14. Let $\Omega_h \cup \Gamma_h$ be the discretization of the unit square $\bar{\Omega}$ and let $\mathcal{L}_h$ be a consistent discretization of the continuous operator $\mathcal{L} = \binom{L}{B}$ (with general elliptic operator $L = -\sum a_{ij}\partial^2/(\partial x_i \partial x_j) + \sum b_i \partial/\partial x_i + c$ where a_{ij}, b, c are continuous, $c \geq 0$, (a_{ij}) positive definite on $\bar{\Omega}$ and B being the trace operator on $\partial\Omega$). Show that, for h sufficiently small, there exists a vector $y \gg 0$ such that $\mathcal{L}_h y \gg 0$ and that there are h–independent constants C, μ so that

$$\|y\|_{h,\infty} \leq C < \infty, \qquad \min_i(\mathcal{L}_h y)_i \geq \mu > 0.$$

Hint: Use $v(x) = \beta - \exp(-\alpha|x - x_0|^2)$.

Exercise 2.15. Describe the matrix $\mathcal{L}_h$ which results from the standard discretization of the Poisson problem on the unit square with Neumann boundary conditions, if half–sided (respectively central) differences are used for the normal derivatives. Show that there is no vector $y \gg 0$ such that $\mathcal{L}_h y \gg 0$.

Exercise 2.16. Implement the finite difference scheme presented in the lecture for one–dimensional equations using the SOR algorithm.

a) Apply it to $-u''(x) = \sin(x)$ on $(0, \pi)$ with $u(0) = u(\pi) = 0$ for several discretizations and compare with the exact solution. Give a doubly logarithmic plot of the error versus h to numerically validate second order convergence.
b) Apply it to $-u''(x) + bu'(x) = 1$ on $(0, 1)$ with $u(0) = u(1) = 0$ and b constant. Check, whether dominating convection $b \gg 1$ influences the speed of convergence of the SOR algorithm. Plot the number of cycles necessary to reach a prescribed error compared with the exact solution for different values of b. Repeat this for different values of h.

Exercise 2.17. Construct fundamental solutions for the operators $\Delta + 1$ and $\Delta - 1$ on $\mathbb{R}^3$. (Hint: Look for a spherically symmetric function of the form $u(x) = f(|x|) = \hat{f}(|x|)/|x|$ with a nice behavior at infinity.)

a) Extend the result to the case $\Delta \pm \omega^2$, $\omega > 0$ by a simple transformation.
b) Show that the so called *Helmholtz operator* $\Delta + \omega^2$ naturally appears if the wave equation $\partial^2 u/\partial^2 t = \Delta u$ is solved by Fourier transformation (in the time variable t).
c) Show that a fundamental solution of the Laplacian is recovered in the case $\omega = 0$.

Exercise 2.18. Construct a Green's function for the Dirichlet Poisson problem

$$\begin{aligned} \Delta u &= f \qquad &&\text{in } \Omega \\ u &= 0 \qquad &&\text{on } \partial\Omega \end{aligned}$$

with the help of the method of images if Ω is the quarter plane

$$\Omega = \{x \in \mathbb{R}^2 \, : \, x_1, x_2 > 0\}.$$

How does the Green's function for the corresponding problem with Neumann conditions ($\partial u/\partial n = 0$ on $\partial\Omega$) look like?

Exercise 2.19. Construct a Green's function for the Dirichlet Poisson problem

$$\begin{aligned} \Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

with the help of the method of images if Ω is the unit ball in $\mathbb{R}^n$. (Hint: use the reflection $\hat{x} = x/\|x\|^2$ and a suitable scaling of the reflected fundamental solution.) Show that the Poisson kernel $P(x,y) = n(y) \cdot \nabla_y G(x,y)$ for $y \in \partial B_1(0)$ is the one used in Theorem 1.14.