CHAPTER 3

Weak solutions

By generalizing the classical solution concept it is possible to treat a larger class of problems (i.e. step functions as data, complicated domains). Moreover, since less regularity is required of the solution, existence proofs are simplified. Finally, numerical methods can be constructed which converge with less assumptions on the regularity of the solution as required, for example, for finite difference schemes. Of course, we impose the natural condition that generalized solutions coincide with classical solutions whenever the latter exist (otherwise, the adjective *generalized* would not be justified). In many cases, however, the generalized solution does not have enough regularity to be classical.

We remark that weak formulations of elliptic problems have already been encountered in Section 2.8 where we considered, for example, $\Delta u = f$ in the sense of distributions on Ω. The idea is to multiply the equation with a test function $\phi \in C_0^\infty(\Omega)$ and integrate by parts (or rather to take the dual product and apply the definition of distributional derivatives). This gives rise to the weak formulation of the Poisson equation

$$\langle u, \Delta\phi \rangle = \langle f, \phi \rangle , \qquad \forall \phi \in \mathscr{D}(\Omega).$$

The formulation is really a generalization because it reduces to the classical one if the function u is sufficiently smooth. However, it is not at all clear, how boundary conditions like $u = 0$ on $\partial\Omega$ can be interpreted in the distributional setup. A possible way out is to restrict the class of distributions which are considered as possible solutions. A very much investigated family of subclasses of $\mathscr{D}'(\Omega)$ are the so called *Sobolev spaces* which turn out to be natural function spaces to be used for the weak formulation of elliptic problems. In the appendix, we briefly introduce these spaces and present a few concepts which are of importance when dealing with boundary conditions.

3.1. Introductory examples

3.1.1. The Dirichlet Helmholtz problem. We start our considerations with the *Dirichlet Helmholtz problem*

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

To obtain a weak formulation of (3.1) we first use a formal argument, i.e. we assume sufficient regularity of all participating objects so that all steps of the derivation are justified.

Multiplying the Helmholtz equation by a function v and integrating over Ω yields

$$-\int_\Omega v\Delta u\,dx + \int_\Omega vu\,dx = \int_\Omega vf\,dx.$$

Using the Gauß theorem for the first integral then leads to

$$\int_\Omega \nabla v\cdot\nabla u\,dx + \int_\Omega vu\,dx = \int_\Omega vf\,dx + \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,d\sigma.$$

Now, we observe that the left hand side is nothing but the $H^1(\Omega)$ scalar product and that the integral involving f is the $\mathbb{L}^2$–product

$$\langle u,v\rangle_1 = \langle f,v\rangle_0 + \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,d\sigma. \tag{3.2}$$

To get rid of the boundary integral, we assume that v vanishes on $\partial\Omega$. Also, the Dirichlet boundary condition on u is incorporated by the requirement $Bu = 0$. By choosing u and v from spaces for which (3.2) makes sense, we end up with the following generalization of (3.1):

Definition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in \mathbb{L}^2(\Omega)$. A function $u \in H_0^1(\Omega)$ is called* weak solution *of* (3.1) *if*

$$\langle u,v\rangle_1 = \langle f,v\rangle_0 \qquad \forall v \in H_0^1(\Omega).$$

If we assume that Ω is a bounded $C^{0,1}$ domain, then the derivation already indicates that classical solutions of the Helmholtz problem are automatically weak solutions. It is important to note, however, that the converse is also true if the weak solution is regular enough. To see this, we assume that $u \in C^2(\Omega)\cap C(\bar\Omega)$ is a weak solution. Since, by definition, $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ we have

$$\langle -\Delta u + u - f, \phi\rangle = 0, \qquad \forall \phi \in C_0^\infty(\Omega)$$

which means $-\Delta u + u = f$ in the sense of distributions. Due to the continuity of $-\Delta u + u$ this implies $f \in C(\Omega)$ and $-\Delta u + u = f$ point wise. The Dirichlet boundary condition is recovered since u is, by assumption, an element of $H_0^1(\Omega)\cap C(\bar\Omega)$ (see Theorem C.4).

If we want to treat more general Dirichlet conditions $u = g$ on $\partial\Omega$ we use a standard trick: we assume the existence of some function u_0 on Ω which has the correct boundary values $u_0 = g$ on $\partial\Omega$. Then, $u - u_0$ has again zero boundary values and satisfies the equation

$$(\Delta+1)(u-u_0) = f - (\Delta+1)u_0 = \hat f$$

which is again of Helmholtz type. For the weak solutions, we proceed exactly in the same way. To guarantee that there is some function u_0 on Ω which has g as boundary values we require that $g \in B(H^1(\Omega))$ (which is $H^{\frac{1}{2}}(\partial\Omega)$

under suitable assumptions on $\partial\Omega$). Then, there exists $u_0 \in H^1(\Omega)$ such that $Bu_0 = g$ and $u - u_0$ should satisfy

$$\langle u - u_0, v\rangle_1 = \langle f, v\rangle_0 - \langle u_0, v\rangle_1 \qquad \forall v \in H_0^1(\Omega).$$

Definition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{0,1}$ domain, $f \in \mathbb{L}^2(\Omega)$ and let $g \in \mathbb{L}^2(\partial\Omega)$ be the trace of some function $u_0 \in H^1(\Omega)$. Then, $u \in H^1(\Omega)$ is called weak solution of $-\Delta u + u = f$ in Ω and $u = g$ on $\partial\Omega$ if $\hat{u} = u - u_0$ satisfies*

$$\langle \hat{u}, v\rangle_1 = \langle f, v\rangle_0 - \langle u_0, v\rangle_1 \qquad \forall v \in H_0^1(\Omega).$$

3.1.2. The Neumann Helmholtz problem. We again consider the Helmholtz equation but this time with a Neumann condition

$$\begin{aligned} -\Delta u + u &= f \qquad &&\text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 &&\text{on } \partial\Omega \end{aligned} \tag{3.3}$$

Again, multiplying with a test function and integration by parts yields (3.2) where now the boundary integral vanishes regardless of the trace of v. This leads to the following generalization of (3.3):

Definition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in \mathbb{L}^2(\Omega)$. A function $u \in H^1(\Omega)$ is called* weak solution *of* (3.3) *if*

$$\langle u, v\rangle_1 = \langle f, v\rangle_0 \qquad \forall v \in H^1(\Omega).$$

Again, if Ω has a C^1 boundary (so that normal derivatives can be defined), any classical solution of (3.3) is also a weak solution. If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a weak solution, we find again $-\Delta u + u = f$ point wise by taking test functions $v \in C_0^\infty(\Omega) \subset H^1(\Omega)$. For $\phi \in C_0^\infty(\mathbb{R}^n)$ we get, since $\phi|_\Omega \in H^1(\Omega)$,

$$\begin{aligned} 0 &= \langle u, \phi\rangle_1 - \langle f, \phi\rangle_0 \\ &= \int_\Omega \nabla u \cdot \nabla\phi\, dx + \int_\Omega u\phi\, dx - \int_\Omega f\phi\, dx \\ &= \int_\Omega (-\Delta u + u - f)\phi\, dx + \int_{\partial\Omega} \phi \frac{\partial u}{\partial n}\, d\sigma. \end{aligned}$$

Since $-\Delta u + u - f = 0$ point wise, we obtain

$$\int_{\partial\Omega} \phi \frac{\partial u}{\partial n}\, d\sigma = 0 \qquad \forall \phi \in C_0^\infty(\mathbb{R}^n). \tag{3.4}$$

Since the restriction of $C_0^\infty(\mathbb{R}^n)$ functions to Ω is dense in $H^1(\Omega)$ (see Theorem C.3) and since the boundary values of $H^1(\Omega)$ functions are dense in $\mathbb{L}^2(\partial\Omega)$ (Theorem C.4), we conclude that $\partial u/\partial n$ is orthogonal to a dense subset of $\mathbb{L}^2(\partial\Omega)$ and thus has to be zero.

We remark that the weak formulation in Definition 3.3 does neither require the existence of an outward normal on $\partial\Omega$ nor $u \in H^2(\Omega)$ so that $\partial u/\partial n$ is, in general, not well defined. Nevertheless, if boundary regularity of the weak solution can be shown, our previous argument proves that the Neumann condition is satisfied.

As for the Dirichlet problem, let us also consider the case of non–homogeneous Neumann conditions $\partial u/\partial n = g$ on $\partial\Omega$. Repeating the derivation, we again arrive at (3.2) where now the Neumann condition is inserted giving rise to the boundary integral

$$\int_{\partial\Omega} v \frac{\partial u}{\partial n}\, d\sigma = \int_{\partial\Omega} vg\, d\sigma = \langle g, Bv\rangle_{0,\partial\Omega}\,.$$

Definition 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{0,1}$ domain, $f \in \mathbb{L}^2(\Omega)$ and let $g \in \mathbb{L}^2(\partial\Omega)$. Then, $u \in H^1(\Omega)$ is called weak solution of $-\Delta u + u = f$ in Ω and $\partial u/\partial n = g$ on $\partial\Omega$ if*

$$\langle \hat{u}, v\rangle_1 = \langle f, v\rangle_0 + \langle g, Bv\rangle_{0,\partial\Omega} \qquad \forall v \in H^1(\Omega).$$

3.1.3. Existence, Uniqueness, Continuous Dependence – the Riesz Lemma. Since the weak formulations of the Helmholtz problem with Dirichlet and Neumann conditions are very similar, we consider the following abstract problem:

Given a Hilbert space V and an element of the dual $l \in V'$, find $u \in V$ such that

$$\langle u, v\rangle_V = l(v) \qquad \forall v \in V. \tag{3.5}$$

We have the important result

Theorem 3.5. *Problem (3.5) admits a unique solution $u \in V$ which depends continuously on the data, i.e. $\|u\|_V = \|l\|_{V'}$.*

Proof: The proof is a simple application of the Riesz representation Theorem (which will be proven below for completeness). ∎

By taking $V = H_0^1(\Omega)$, we recover the Dirichlet and with $V = H^1(\Omega)$ the Neumann problem. In the homogeneous cases (Definitions 3.1 and 3.3) the linear functional l is given by

$$l(v) = \langle f, v\rangle_0\,, \qquad f \in \mathbb{L}^2(\Omega).$$

We see here, that we have been unnecessarily restrictive in deriving a weak formulation. Instead of requiring $f \in \mathbb{L}^2(\Omega)$ we could as well have assumed $f \in H^{-1}(\Omega)$ (the dual of $H_0^1(\Omega)$) for the Dirichlet and $f \in H^1(\Omega)'$ for the Neumann problem. Then, the integral $\langle f, v\rangle_0$ is just interpreted as a linear functional acting on v (as in the theory of generalized functions). As far as f is concerned, the same holds also for the non–homogeneous boundary conditions. In the Dirichlet case, the linear functional is

$$l(v) = \langle f, v\rangle_0 - \langle u_0, v\rangle_1$$

and for the Neumann problem

$$l(v) = \langle f, v \rangle_0 + \langle g, Bv \rangle_{0,\partial\Omega} . \tag{3.6}$$

Note that $v \mapsto \langle g, Bv \rangle_{0,\partial\Omega}$ is a continuous linear functional on $H^1(\Omega)$ since $B : H^1(\Omega) \mapsto \mathbb{L}^2(\partial\Omega)$ is a bounded linear mapping. We also remark that the condition $g \in \mathbb{L}^2(\partial\Omega)$ in the Neumann case could be relaxed if we used that $B(H^1(\Omega)) = H^{\frac{1}{2}}(\partial\Omega)$ has higher regularity than $\mathbb{L}^2(\partial\Omega)$. Indeed, $\langle g, Bv \rangle_{0,\partial\Omega}$ could be replaced by a linear functional $\hat{l}$ from the dual of $H^{\frac{1}{2}}(\partial\Omega)$ acting on Bv without loosing the property that $v \mapsto \hat{l}(Bv)$ is a continuous linear functional on $H^1(\Omega)$.

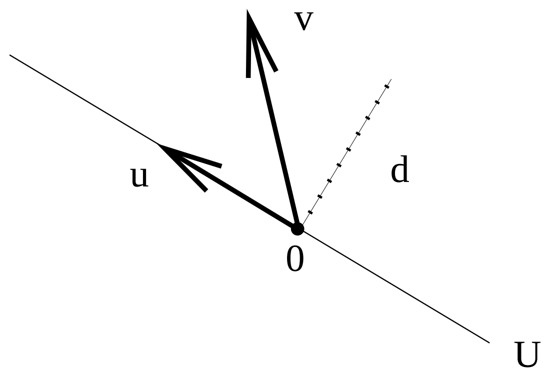
For completeness, we prove the Riesz representation theorem. It will also give a geometrical interpretation of the weak solution concept in Hilbert spaces. We begin with the important *Projection Theorem.*

Theorem 3.6. *Let V be a Hilbert space and let U be a closed subspace of V. Then every $v \in V$ has a unique decomposition $v = u + w$, where $u \in U$ and $w \in U^\perp$. ($U^\perp$ is the orthogonal complement of U, i.e. the set of all vectors which are orthogonal to U.)*

Proof: At the core of the proof is the so called *parallelogram law* which relates the length of the edges a, b of a parallelogram with those of the diagonals

$$\frac{1}{2}\left(\|a+b\|^2 + \|a-b\|^2\right) = \|a\|^2 + \|b\|^2.$$

(We skip the subscript V at the norm and the scalar product.) Motivated by finite dimensional Euclidean geometry, we expect that the orthogonal projection of $v \in V$ on to the subspace U is given by the element with minimal distance to v.



We define

$$d := \inf_{x \in U} \|x - v\|$$

as minimal distance and pick a minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset U$ such that $d_n = \|u_n - v\| \to d$. Applying the parallelogram law with $a = u_n - v$ and

$b = u_m - v$ we get

$$\begin{aligned}\|u_n - u_m\|^2 &= 2\|u_n - v\|^2 + 2\|u_m - v\|^2 - \|u_n + u_m - 2v\|^2 \\ &= 2d_n^2 + 2d_m^2 - 4\|\frac{u_n + u_m}{2} - v\|^2.\end{aligned}$$

Since $(u_n + u_m)/2 \in U$, we have by definition that

$$\|\frac{u_n + u_m}{2} - v\| \geq d$$

and thus

$$\|u_n - u_m\|^2 \leq 2(d_n^2 - d^2) + 2(d_m^2 - d^2)$$

which shows that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since U is closed, the sequence converges to an element $u \in U$ which satisfies $\|u - v\| = d$. Consequently, for any $x \in U$, the function

$$f(t) = \|u + tx - v\|^2, \qquad t \in \mathbb{R}$$

has a minimum at $t = 0$. The condition $f'(0) = 0$ then yields

$$\langle v - u, x\rangle = 0, \qquad \forall x \in U \tag{3.7}$$

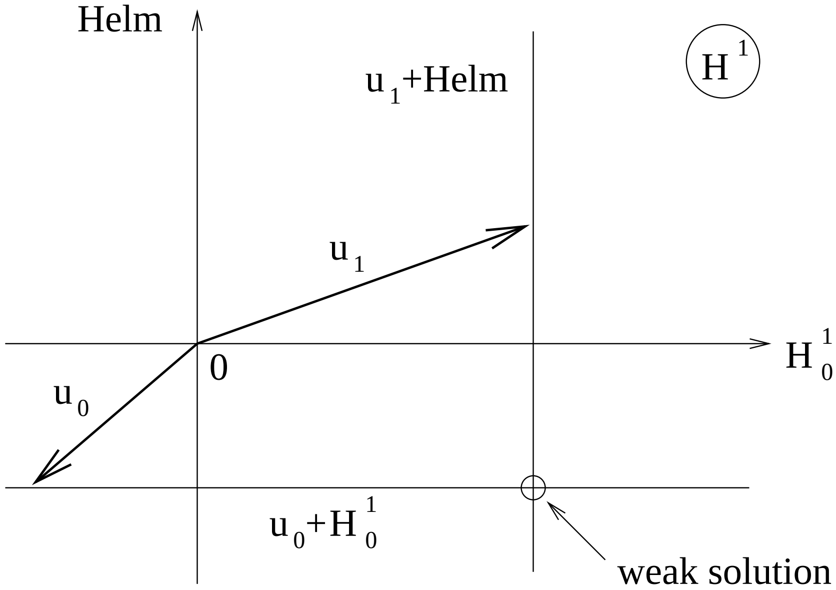
which shows that $w = v - u$ is an element of $U^\perp$.

Uniqueness of the decomposition follows easily from (3.7). If $\hat{u}$ is another vector such that $\langle v - \hat{u}, x\rangle = 0$ for all $x \in U$, the difference with (3.7) yields

$$\langle u - \hat{u}, x\rangle = 0 \qquad \forall x \in U.$$

In particular, for $x = u - \hat{u}$ we find $\|u - \hat{u}\|^2 = 0$ which implies $u = \hat{u}$. ■

Now, we can geometrically interpret the concept of weak solutions for the Helmholtz problem with general Dirichlet boundary conditions. If $u_0 \in H^1(\Omega)$ has the required boundary values g, the weak solution u is restricted to the affine linear subspace $u_0 + H_0^1(\Omega)$ which contains all functions in $H^1(\Omega)$ which have g as trace.



Conversely, if we just consider the Helmholtz equation $-\Delta u + u = f$ without boundary conditions, again the set of solutions makes up an affine linear subspace of $H^1(\Omega)$. If u_1 is a special solution then the solution manifold is $u_1 + \mathrm{Helm}(\Omega)$ where $\mathrm{Helm}(\Omega)$ contains all homogeneous solutions $-\Delta u + u = 0$. In the weak formulation, boundary conditions are neglected by using test functions which are compactly supported in Ω. Thus

$$\mathrm{Helm}(\Omega) = \left\{ u \in H^1(\Omega) \,:\, -\Delta u + u = 0 \quad \text{in } \mathscr{D}'(\Omega) \right\}. \tag{3.8}$$

It is easy to see that $\mathrm{Helm}(\Omega)$ is really the orthogonal complement of $H_0^1(\Omega)$ as indicated in the figure. If $u \in H_0^1(\Omega)^\perp$, then $\langle u, \phi \rangle_1 = 0$ for all $\phi \in \mathscr{D}(\Omega)$ since, by definition, test functions are dense in $H_0^1(\Omega)$. An integration by parts then yields $\langle -\Delta u + u, \phi \rangle = 0$ in the sense of distributions so that $u \in \mathrm{Helm}(\Omega)$. Conversely, if $u \in \mathrm{Helm}(\Omega)$ then u is, by definition, orthogonal to a dense subset of $H_0^1(\Omega)$ and thus to $H_0^1(\Omega)$ itself.

We conclude that the space $H^1(\Omega)$ is the orthogonal sum of two closed subspaces $H_0^1(\Omega)$ and $\mathrm{Helm}(\Omega)$. One of these subspaces takes care of the boundary conditions, the other one of the partial differential equation. A specific problem gives rise to affine linear spaces $u_0 + H_0^1(\Omega)$ and $u_1 + \mathrm{Helm}(\Omega)$. The first one contains all elements with the correct boundary value problems, the second one is the collection of all solutions of the (inhomogeneous) PDE. The solution of the boundary value problem is then naturally given by the intersection of the two spaces which is a single point (due to orthogonality). In other words, the weak solution is the projection of $u_1 + \mathrm{Helm}(\Omega)$ onto the correct boundary values $u_0 + H_0^1(\Omega)$ (or, vice versa, the projection of $u_0 + H_0^1(\Omega)$ on the solution space $u_1 + \mathrm{Helm}(\Omega)$). Geometrically (in 2D) it is clear that a slight change of u_0 and u_1 (the data of the problem) only changes the intersection point (the solution) a little bit. This is a geometrical manifestation of the continuous dependence on the data.

Finally, let us use the projection theorem to prove the Riesz representation theorem.

Theorem 3.7. *The dual space of a Hilbert space is isometric to the Hilbert space itself. In particular, for every $u \in V$ the linear functional on V defined by*

$$l_u(v) := \langle u, v \rangle_V$$

is bounded with norm $\|l_u\|_{V'} = \|u\|_V$. Moreover, for every $l \in V'$, there exists a unique $u \in V$ such that

$$l(v) = \langle u, v \rangle_V \qquad \forall v \in V,$$

and, furthermore, $\|u\|_V = \|l\|_{V'}$.

Proof: Boundedness of l_u is obtained with the Cauchy–Schwarz inequality

$$|l_u(v)| = |\langle u, v \rangle| \leq \|u\| \, \|v\|$$

which shows that $\|l_u\| \leq \|u\|$. Equality follows upon taking $v = u$. (We again neglect subscripts on norms and scalar products.) Conversely, if $l \in V'$ is given, we define $U = \ker l$ which is a closed subspace of V. For $l \neq 0$ (otherwise $u = 0$ is the unique representation), $U^\perp$ is one dimensional. Indeed, if w_1, w_2 are vectors in $U^\perp$ then

$$l(w_2)w_1 - l(w_1)w_2 \in U \cap U^\perp = \{0\}$$

which shows that w_1 and w_2 are linearly dependent. (Here we have used the projection theorem which shows that V is the orthogonal sum of U and $U^\perp$ if U is a closed subspace.) Now, let $\hat{u} \in U^\perp$ with $\|\hat{u}\| = 1$. Then for $v \in V$

$$v = \langle \hat{u}, v \rangle \hat{u} + (v - \langle \hat{u}, v \rangle \hat{u})$$

is an orthogonal decomposition with $\langle \hat{u}, v \rangle \hat{u} \in U^\perp$ so that $v - \langle \hat{u}, v \rangle \hat{u} \in U = \ker l$ which gives

$$l(v) = \langle \hat{u}, v \rangle l(\hat{u}) = \langle l(\hat{u})\hat{u}, v \rangle .$$

Setting $u = l(\hat{u})\hat{u}$ we thus have $l = l_u$. The vector u is clearly unique for if $\bar{u}$ is another such vector then

$$\langle \bar{u} - u, v \rangle = 0 \qquad \forall v \in V$$

which implies $\bar{u} = u$ (by setting $v = \bar{u} - u$). We have already seen that $\|l_u\| = \|u\|$ which completes the proof. ∎

The isometry which assigns to a functional the corresponding Riesz representation is called *Riesz operator.*

Corollary 3.8. *Let V be a Hilbert space. There exists an isometry $\mathcal{R} : V' \mapsto V$ (i.e. $\mathcal{R}$ and $\mathcal{R}^{-1}$ are bounded linear operators with $\|\mathcal{R}l\| = \|l\|$ for all $l \in V'$) which is uniquely determined by the condition*

$$\langle \mathcal{R}l, v \rangle_V = l(v) = \langle l, v \rangle_{V' \times V} \qquad \forall v \in V, \, l \in V'.$$

$\mathcal{R}$ is called Riesz operator.

3.1.4. The Dirichlet Poisson problem. In the previous example we have seen how problems with inhomogeneous Dirichlet conditions can be reduced to the homogeneous case. Therefore, we will only consider

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{3.9}$$

As before, the weak formulation is obtained by multiplying the equation with a test function and integrating by parts. We find

$$\langle \nabla u, \nabla v \rangle_0 = \langle f, v \rangle_0 + \int_{\partial\Omega} v \frac{\partial u}{\partial n}\, d\sigma.$$

Taking u and v from $H_0^1(\Omega)$ we obtain

Definition 3.9. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in \mathbb{L}^2(\Omega)$. A function $u \in H_0^1(\Omega)$ is called* weak solution *of* (3.9) *if*

$$\langle \nabla u, \nabla v \rangle_0 = \langle f, v \rangle_0 \qquad \forall v \in H_0^1(\Omega).$$

In order to apply Theorem 3.5 we first have to show that $V = H_0^1(\Omega)$ together with the scalar product $\langle u, v \rangle_V = \langle \nabla u, \nabla v \rangle_0$ is a Hilbert space (i.e. that $\langle u, u \rangle_V = 0$ implies $u = 0$ – and that V Cauchy sequences converge in the V norm). This is accomplished with the well known Poincaré inequality. An application of Theorem 3.5 then shows that the weak formulation of (3.9) admits a unique solution which depends continuously on the data if Ω is bounded in at least one direction.

Theorem 3.10. **(Poincaré inequality)**
Let $\Omega \subset \mathbb{R}^n$ be an open set which can be enclosed between two parallel hyperplanes of distance $d > 0$. Then, the $H^m(\Omega)$ seminorm

$$|u|_{m,2} = \left(\sum_{|\alpha|=m} \langle \nabla^\alpha u, \nabla^\alpha u \rangle_0 \right)^{\frac{1}{2}}$$

is equivalent to $\|\cdot\|_{m,2}$ on $H_0^m(\Omega)$. In particular, there exists a constant $C > 0$ which just depends on m and d such that

$$C\|u\|_{m,2} \le |u|_{m,2} \le \|u\|_{m,2} \qquad \forall u \in H_0^m(\Omega)$$

and $H_0^m(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle := \sum_{|\alpha|=m} \langle \nabla^\alpha u, \nabla^\alpha v \rangle_0 \,.$$

Proof: We assume, without loss of generality, that Ω is contained in the strip $\{x \in \mathbb{R}^n \,:\, 0 \le x_1 \le d\}$. For $u \in C_0^\infty(\Omega)$ we obtain with integration by parts

$$\|u\|_{0,2}^2 = \int_\Omega 1 \cdot u^2 \, dx = -\int_\Omega x_1 2u \frac{\partial u}{\partial x_1}\, dx \le 2d \int_\Omega |u| \left| \frac{\partial u}{\partial x_1} \right| dx.$$

Using Cauchy–Schwarz inequality

$$\|u\|_{0,2}^2 \leq 2d\|u\|_{0,2}\|\frac{\partial u}{\partial x_1}\|_{0,2}$$

and after dividing by $\|u\|_{0,2}$

$$\|u\|_{0,2} \leq 2d\|\frac{\partial u}{\partial x_1}\|_{0,2}.$$

Applying this result to $\nabla^\beta u$ with $|\beta| < m$ and $u \in H_0^m(\Omega)$ we get (since $\nabla^\beta u \in H_0^1(\Omega)$)

$$\|\nabla^\beta u\|_{0,2} \leq 2d\|\nabla^{\beta+e_1} u\|_{0,2}.$$

Then, by repeating the argument for $\beta + e_1$ (if $|\beta| + 1 < m$) we arrive at

$$\|\nabla^\beta u\|_{0,2} \leq (2d)^{m-|\beta|}\|\nabla^{\beta+(m-|\beta|)e_1} u\|_{0,2} \leq (2d)^{m-|\beta|}|u|_{m,2}.$$

Summing up over all $|\beta| < m$ then yields

$$\|u\|_{m,2}^2 \leq \left(1 + \sum_{|\beta|<m} (2d)^{2(m-|\beta|)}\right) |u|_{m,2}^2$$

which completes the proof since the other estimate $|u|_{m,2} \leq \|u\|_{m,2}$ is trivial. ■

3.1.5. The Neumann Poisson problem. Repeating the derivation of the weak formulation, we find for

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega \end{aligned} \tag{3.10}$$

Definition 3.11. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in \mathbb{L}^2(\Omega)$. A function $u \in H^1(\Omega)$ is called* weak solution *of* (3.10) *if*

$$\langle \nabla u, \nabla v \rangle_0 = \langle f, v \rangle_0 \qquad \forall v \in H^1(\Omega).$$

We remark that for bounded Ω the weak solution is not unique. If $u \in H^1(\Omega)$ is a solution, the same holds for $u + c \in H^1(\Omega)$ where c is any constant. Consequently, $V = H^1(\Omega)$ together with $\langle u, v \rangle_V = \langle \nabla u, \nabla v \rangle_0$ cannot be a Hilbert space since otherwise the solution would be unique due to Theorem 3.5. Indeed, $\langle u, u \rangle_V$ vanishes exactly on the subspace U of constant functions if Ω is connected (see Theorem B.9; if Ω is not connected, the critical functions are those which are constant on each connected component with not necessarily the same constants). To fix this arbitraryness in the solution we should therefore work in $U^\perp$. If Ω is bounded, then the constant function $e(x) = 1$ is an element of $V = H^1(\Omega)$ and, for connected Ω, the projection on U is given by

$$P_U(v) = \langle e, v \rangle_V e = (\langle \nabla e, \nabla v \rangle_0 + \langle e, v \rangle_0)e = \int_\Omega v \, dx \, e$$

Consequently, $U^\perp$ can be characterized as

$$U^\perp = \left\{ u \in H^1(\Omega) \ : \ \int_\Omega u\,dx = 0 \right\}.$$

To show that $U^\perp$ with $\langle \cdot, \cdot \rangle_V$ is a Hilbert space we need a generalization of the Poincaré inequality.

Theorem 3.12. (Bramble–Hilbert)
Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected $C^{0,1}$ domain and let $m \in \mathbb{N}$, $1 \le p < \infty$. Assume further that $\{\mu_1, \dots, \mu_k\}$ is a family of continuous seminorms on $W^{m,p}(\Omega)$ with the property that $\mu_i(P) = 0$ for all polynomials of order less than m and all $i = 1, \dots, k$ implies $P \equiv 0$. Then, the norm

$$\|u\|_\mu = \left(\sum_{|\alpha|=m} \|\nabla^\alpha u\|^p_{\mathbb{L}^p(\Omega)} + \sum_{i=1}^k \mu_i(f)^p \right)^{\frac{1}{p}}$$

is equivalent to the $W^{m,p}(\Omega)$ norm.

Proof: Continuity of μ_i implies for some constant C_i

$$\mu_i(u) \le C_i \|u\|_{m,p}, \qquad \forall u \in W^{m,p}(\Omega)$$

so that $\|u\|_\mu \le C\|u\|_{m,p}$ is easy. A converse inequality is shown by contradiction. We assume that there is a sequence $(u_l)_{l\in\mathbb{N}} \subset W^{m,p}(\Omega)$ such that $\|u_l\|_{m,p} = 1$ but $\|u_l\|_\mu < 1/l$. The latter condition implies

$$\lim_{l\to\infty} \|\nabla^\alpha u_l\|_{\mathbb{L}^p(\Omega)} = 0, \qquad \forall |\alpha| = m, \tag{3.11}$$

and

$$\lim_{l\to\infty} \mu_i(u_l) = 0, \qquad \forall i = 1, \dots, k. \tag{3.12}$$

Since $W^{m,p}(\Omega)$ is compactly embedded in $W^{m-1,p}(\Omega)$ (see Theorem C.2), the $\|\cdot\|_{m,p}$ bounded sequence $(u_l)_{l\in\mathbb{N}}$ has a convergent subsequence in $W^{m-1,p}(\Omega)$ (which we denote again $(u_l)_{l\in\mathbb{N}}$ to avoid subscripts). If u denotes the limit of this subsequence, we have $\nabla^\alpha u = 0$. Indeed, for any $\phi \in \mathscr{D}(\Omega)$ and $|\alpha| = m$ we have

$$\langle \nabla^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \nabla^\alpha \phi \rangle$$

and since $u_l \to u$ in $\mathbb{L}^p(\Omega)$ and thus also in $\mathscr{D}'(\Omega)$

$$\langle \nabla^\alpha u, \phi \rangle = (-1)^{|\alpha|} \lim_{l\to\infty} \langle u_l, \nabla^\alpha \phi \rangle = \lim_{l\to\infty} \langle \nabla^\alpha u_l, \phi \rangle .$$

Due to (3.11) we thus find

$$\langle \nabla^\alpha u, \phi \rangle = 0 \qquad \forall \phi \in \mathscr{D}(\Omega),\ |\alpha| = m$$

so that $\|u - u_l\|_{m,p} \to 0$ for $l \to \infty$. Since u satisfies $\nabla^\alpha u = 0$ for all $|\alpha| = m$, Theorem B.9 implies that u is a polynomial of degree less than m. Now (3.12) shows (due to continuity of μ_i) that $\mu_i(u) = 0$ for all i so that by assumption $u = 0$. This however is in contradiction to the assumption $\|u_l\|_{m,p} = 1$ for all l since $\lim_{l\to\infty} \|u_l\|_{m,p} = \|u\|_{m,p} = 0$. ∎

Now, we are in the position to prove existence and uniqueness for the Neumann Poisson problem on a bounded domain.

Theorem 3.13. *Let $\Omega \subset \mathbb{R}^n$ be bounded and connected with a $C^{0,1}$ boundary. If $f \in \mathbb{L}^2(\Omega)$ satisfies $\langle 1, f \rangle_0 = 0$ then problem (3.10) admits a solution which is unique under the condition $\langle 1, u \rangle_0 = 0$. Furthermore, we have $\|u\|_{1,2} \le C\|f\|_{0,2}$ for some constant C which is independent of u and f.*

Proof: We consider the Hilbert space

$$V = \{\, u \in H^1(\Omega) \,:\, \langle 1, u \rangle_0 = 0 \,\}$$

with the scalar product $\langle u, v \rangle_V = \langle \nabla u, \nabla v \rangle_0$. On V, the corresponding norm is equivalent to the $H^1(\Omega)$ norm and thus V is a Hilbert space with this scalar product (just apply Theorem 3.12 with the seminorm $\mu_1(u) = |\langle 1, u \rangle_0|$). Then Theorem 3.5 yields a unique solution u to the weak formulation of (3.10) whose norm can be estimated by the norm of the functional

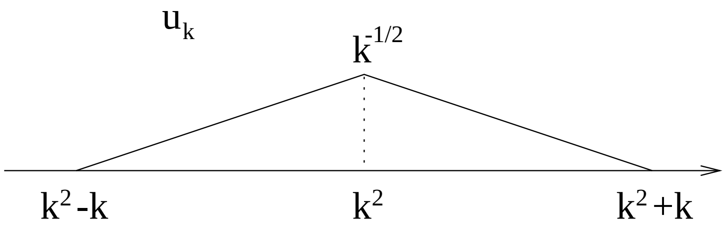
$$l_f(v) = \langle f, v \rangle_0 \qquad v \in V$$

in the dual space V'. (Note that the condition $\langle 1, f \rangle_0 = 0$ ensures that $f \in V$ and thus $l_f \in V'$.) By definition of the dual norm, we have

$$\|l_f\|_{V'} = \sup_{\|v\|_V = 1} l_f(v) = \sup_{\|v\|_V = 1} \langle f, v \rangle_0 \le \sup_{\|v\|_V = 1} \|f\|_{0,2}\,\|v\|_{0,2} \le \|f\|_{0,2}.$$

■

If Ω is not connected, one can apply the same considerations to each of the connected components. In the case of unbounded Ω, uniqueness of the solution is no problem if the constant function $e \equiv 1$ is not contained in $H^1(\Omega)$. Indeed, $\nabla u = 0$ implies $u = const$ (Theorem B.9) which again implies $u = 0$ (which is, by assumption, the only constant function in $H^1(\Omega)$). However, Theorem 3.12 cannot be applied directly in that case since boundedness of Ω is important in the proof to ensure that $H^1(\Omega)$ is compactly embedded in $\mathbb{L}^2$. To see this, consider the functions $u_k \in H^1(0, \infty)$ given by the figure below.



It is easily checked that $\|\nabla u_k\|_{0,2} = \mathcal{O}(1/k)$ while $\|u_k\|_{0,2} = \mathcal{O}(1)$. Hence, the H^1 norm can never be controled by the norm of the gradient alone. For such cases one has to require a certain decay of the functions and consider weighted Sobolev spaces instead of $H^1(\Omega)$ as starting point.