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Tutorial *Analytical Methods for PDEs*

Return: will be announced
Tutorials can also be found at `http://www.mathematik.uni-kl.de/~junk`

Exercise 1:

Let $\boldsymbol{v} : \mathbb{R}^+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be a smooth vector field (at least C^1) with uniformly bounded derivatives. (Think of the velocity field of a gas or fluid flow.) The solution operator associated to the ODE $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ will be denoted $\boldsymbol{H}(t; \boldsymbol{\xi}, \tau)$, i.e. $t \mapsto \boldsymbol{H}(t; \boldsymbol{\xi}, \tau)$ is the solution for the initial conditions $\boldsymbol{x}(\tau) = \boldsymbol{\xi}$. For $\tau = 0$, we briefly write $\boldsymbol{H}(t; \boldsymbol{\xi}) = \boldsymbol{H}(t; \boldsymbol{\xi}, 0)$. (In other words, $\boldsymbol{H}(t; \boldsymbol{\xi}, \tau)$ tells you the position of a fluid particle at time t which used to be in $\boldsymbol{\xi}$ at time τ.)

Check your ODE knowledge: For every $t \geq 0$, $\boldsymbol{H}(t; \cdot)$ is a smooth, bijective mapping with

$$\frac{\partial \boldsymbol{H}(t; \boldsymbol{\xi})}{\partial t} = \boldsymbol{v}(t, \boldsymbol{\xi}), \qquad \det \left(\frac{\partial X_i(t; \boldsymbol{\xi})}{\partial \xi_j} \right)_{i,j=1,\dots,d} > 0.$$

Remark on **Check**–questions: If you feel unsure about parts of the statement, you should spend some time on reviewing the corresponding material. However, you should not return detailed answers on these questions since they are not of primary concern.

Assume now that $\Omega \subset \mathbb{R}^d$ is a *control volume*, i.e. a bounded, open and connected set with a C^1–boundary. (Note that all fluid particles which are originally ($\tau = 0$) contained in Ω are found in the set $\boldsymbol{H}(t, \Omega)$ at time t.)

Check your analysis knowledge: For every $t \geq 0$, the set $\boldsymbol{H}(t, \Omega)$ is also a control volume.

The actual problem in this exercise is to prove the **transport theorem**: with the assumptions from above, any smooth function $\rho : \mathbb{R}^+ \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfies

$$\frac{d}{dt} \int_{\boldsymbol{H}(t,\Omega)} \rho(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_{\boldsymbol{H}(t,\Omega)} \left. \frac{\partial \rho}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho \boldsymbol{v}) \right|_{(t,\boldsymbol{x})} d\boldsymbol{x}.$$

Exercise 2:

The transport theorem can be used to derive the equations of gas and fluid dynamics.

a) If ρ denotes the density of mass, then the total mass in a control volume Ω is given by the integral of ρ over Ω. A natural assumption on the fluid flow (with velocity field $\boldsymbol{v}$) is mass conservation which states that the mass in a control volume $H(t, \Omega)$ which follows the flow does not change. Use this assumption together with the transport theorem to show that ρ satisfies the equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho \boldsymbol{v}) = 0. \tag{1}$$

b) The total momentum of a control volume $H(t,\Omega)$ is obtained by integrating the momentum density $\rho\boldsymbol{v}$. According to Newton's law, the rate of change of momentum is related to the forces acting on the control volume. We assume that the total force splits into volume forces with density $(\rho\boldsymbol{f})(t,\boldsymbol{x})$ (like gravity) and surface forces described by the symmetric $d\times d$–matrix $\sigma(t,\boldsymbol{x})$ (i.e. $\sigma\boldsymbol{n}dS$ is the force acting on the surface element dS with normal $\boldsymbol{n}$ - like pressure or traction forces). Thus, Newton's law leads to

$$\frac{d}{dt}\int_{\boldsymbol{H}(t,\Omega)} \rho\boldsymbol{v}\,d\boldsymbol{x} = \int_{\boldsymbol{H}(t,\Omega)} \rho\boldsymbol{f}\,d\boldsymbol{x} + \int_{\partial\boldsymbol{H}(t,\Omega)} \sigma\boldsymbol{n}\,dS.$$

(Note: arguments $(t,\boldsymbol{x})$ are suppressed everywhere to keep expressions short.) Using this force balance, show with the transport theorem that

$$\frac{\partial\rho v_i}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho v_i\boldsymbol{v}) = \rho f_i + \sum_{j=1}^{d}\frac{\partial\sigma_{ij}}{\partial x_j}, \qquad i=1,\dots,d. \tag{2}$$

c) The total energy of a control volume is obtained through the energy density ρE. The balance equation for energy involves the densities of power $\rho\boldsymbol{f}\cdot\boldsymbol{v}$ and $(\sigma\boldsymbol{n})\cdot\boldsymbol{v}$ related to the forces acting on the volume. Another source for energy changes is the production of heat in the volume (with density ρQ) and a possible heat-flux $\boldsymbol{q}$ through the surface. Altogether, we have

$$\begin{aligned}\frac{d}{dt}\int_{\boldsymbol{H}(t,\Omega)} \rho E\,d\boldsymbol{x} = \int_{\boldsymbol{H}(t,\Omega)} \rho\boldsymbol{f}\cdot\boldsymbol{v}\,d\boldsymbol{x} &+ \int_{\partial\boldsymbol{H}(t,\Omega)} (\sigma\boldsymbol{n})\cdot\boldsymbol{v}\,dS \\ &+ \int_{\boldsymbol{H}(t,\Omega)} \rho Q\,d\boldsymbol{x} - \int_{\partial\boldsymbol{H}(t,\Omega)} \boldsymbol{q}\cdot\boldsymbol{n}\,dS.\end{aligned}$$

Using again the transport theorem, derive the energy equation

$$\frac{\partial\rho E}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho E\boldsymbol{v}) = \rho\boldsymbol{f}\cdot\boldsymbol{v} + \operatorname{div}_{\boldsymbol{x}}(\sigma\boldsymbol{v}) + \rho Q - \operatorname{div}\boldsymbol{q} \tag{3}$$

Note that (1), (2) and (3) is an under-determined system of equations for $\rho,\boldsymbol{v},E$ since it also involves the quantities $\boldsymbol{f},\sigma,Q,\boldsymbol{q}$. In order to close the system, we need to express the additional quantities in terms of the variables $\rho,\boldsymbol{v},E$. The system of **Euler equations** is obtained by assuming the following closure relations (which are reasonable for gases): $Q=0$, $\boldsymbol{q}=\boldsymbol{0}$, $\sigma_{ij}=-p\delta_{ij}$ where the pressure p is proportional to the product of temperature and density $p=\rho RT$ (R is the universal gas constant). The temperature, on the other hand, is proportional to the internal energy $e=C_VT$ where the constant C_V is the specific heat at constant volume and $e=E-|\boldsymbol{v}|^2/2$. The force density $\boldsymbol{f}$ is typically assumed to be an explicitly given function, like $\boldsymbol{f}=(0,0,-g)$ in the case $d=3$ with the gravity constant g. Similarly, heat sources Q are assumed to be given functions of t and $\boldsymbol{x}$. The **compressible Navier-Stokes equations** are obtained by allowing heat-fluxes $\boldsymbol{q}=-k\nabla T$ (Fourier's law) and taking into account viscous stresses

$$\sigma_{ij} = (-p+\lambda\operatorname{div}_{\boldsymbol{x}}\boldsymbol{v})\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

The remaining quantities are taken as in the Euler case.