Dr. Michael Junk
Dipl. Math. Dirk Kehrwald

SS 2000

Tutorial *Analytical Methods for PDEs*

Return: will be announced

Exercise 3:

The system of Euler equations (in 3D space) derived in Exercise 2 has the form

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho \boldsymbol{v}) = 0 \\
&\frac{\partial \rho v_i}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho v_i \boldsymbol{v}) + \frac{\partial p}{\partial x_i} = 0, \qquad i = 1,2,3 \\
&\frac{\partial \rho E}{\partial t} + \operatorname{div}_{\boldsymbol{x}}((\rho E + p)\boldsymbol{v}) = 0
\end{aligned}$$

where $p = \rho R T$, $e = C_V T$ and $e = E - |\boldsymbol{v}|^2/2$. For convenience, we also introduce $\kappa = R/C_V + 1$. In the following, we assume that $\rho, v_1, v_2, v_3, E \in C^1(\mathbb{R}^+ \times \mathbb{R}^3)$ satisfy the Euler equations as well as $\rho, T > 0$.

a) Show that for any fixed $\rho_0, T_0 > 0$, $S_0 \in \mathbb{R}$, the *entropy* $S = C_V \ln \frac{T/T_0}{(\rho/\rho_0)^{\kappa-1}} + S_0$ satisfies the equation

$$\frac{\partial S}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} S = 0.$$

b) Using the result of (a), show that the system

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho \boldsymbol{v}) = 0 \\
&\frac{\partial \rho v_i}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho v_i \boldsymbol{v}) + \nabla p = 0, \qquad i = 1,2,3 \\
&\frac{\partial S}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} S = 0
\end{aligned}$$

with $p/p_0 = \exp((S - S_0)/C_V)(\rho/\rho_0)^\kappa$ is equivalent to the Euler system (i.e. a solution $\rho, \boldsymbol{v}, S$ can be transformed into a solution $\rho, \boldsymbol{v}, E$ of the Euler system and vice versa).

c) Using the result of (b), show that special solutions of the Euler system can be obtained by solving the *isentropic Euler system*

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho \boldsymbol{v}) = 0 \\
&\frac{\partial \rho v_i}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(\rho v_i \boldsymbol{v}) + C \nabla \rho^\kappa = 0, \qquad i = 1,2,3
\end{aligned}$$

with some $C > 0$.

please turn over

d) Show that the isentropic Euler system is equivalent to the system

$$\begin{aligned}
&\frac{\partial p}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} p + \kappa p \operatorname{div}_{\boldsymbol{x}} \boldsymbol{v} = 0 \\
&\frac{\partial v_i}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} v_i + (p/C)^{-\frac{1}{\kappa}} \nabla p = 0, \qquad i = 1, 2, 3
\end{aligned}$$

e) Linearize the system in (d) around the solution $p = \bar{p}$ and $\boldsymbol{v} = \mathbf{0}$. Use the ansatz $p = \bar{p} + \epsilon \hat{p}$ and $\boldsymbol{v} = \epsilon \hat{\boldsymbol{v}}$ as in Section 1.2.7 and derive approximate linear equations for $\hat{p}, \hat{\boldsymbol{v}}$. Show that $\hat{p}$ solves the wave equation. What is the wave velocity?

And what is all that good for? Well, Euler equations describe the time evolution of gases, for example the air around us (remember, we have just used conservation of mass, momentum and energy as well as some closure relations in the derivation). The state $p = \bar{p}$ and $\boldsymbol{v} = \mathbf{0}$ represents a situation without air movement (like in a closed room) with constant atmospheric pressure. Producing a sound (for example, by talking) now leads to a little pressure deviation $\epsilon \hat{p}$ and this deviation satisfies the wave equation (approximately) – a *sound wave*. Isn't that remarkable? Essentially by using some conservation principles we can explain sound waves. How to solve the wave equation we will see later ...