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SS 2000

Tutorial *Analytical Methods for PDEs*

Return: Tuesday, 02.05. 1pm, metal box (Kehrwald), ground floor bldg. 48

Exercise 4: (something on multi-indices)

- Show that the number of $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ with $|\boldsymbol{\alpha}| = N$ is given by $\binom{d+N-1}{N}$.
- Prove the multi-dimensional binomial formula for $\boldsymbol{a} \in \mathbb{R}^d$
$$\left(\sum_{i=1}^{d} a_i\right)^m = \sum_{|\boldsymbol{\gamma}|=m} \frac{m!}{\boldsymbol{\gamma}!} \boldsymbol{a}^{\boldsymbol{\gamma}}, \qquad \boldsymbol{\gamma}! = \gamma_1! \dots \gamma_d!.$$
- Show the Leibniz formula for infinitely smooth $f, g : \mathbb{R}^d \mapsto \mathbb{R}$
$$\nabla^{\boldsymbol{\alpha}}(fg) = \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \nabla^{\boldsymbol{\beta}} f \nabla^{\boldsymbol{\alpha}-\boldsymbol{\beta}} g, \qquad \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$$
where $\boldsymbol{\alpha} \in \mathbb{N}_0^d$, and $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ abbreviates $\beta_i \leq \alpha_i$ for $i = 1, \dots, d$.
- Use the previous result with $f(\boldsymbol{x}) = \exp(\boldsymbol{x} \cdot \boldsymbol{u})$ and $g(\boldsymbol{x}) = \exp(\boldsymbol{x} \cdot \boldsymbol{v})$ for two fixed vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$ to show that
$$(\boldsymbol{u} + \boldsymbol{v})^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \boldsymbol{u}^{\boldsymbol{\beta}} \boldsymbol{v}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}$$

Exercise 5:

Let $n, N \in \mathbb{N}$ and $I = \{\boldsymbol{\alpha} \in \mathbb{N}_0^n : |\boldsymbol{\alpha}| \leq N\}$. In the following, the coefficients of vectors in $\mathbb{R}^{|I|}$ are indexed with $\boldsymbol{\alpha} \in I$. Based on a function $G : \mathbb{R}^n \times \mathbb{R}^{|I|} \mapsto \mathbb{R}$, we consider the partial differential equation $G\left(\boldsymbol{y}, (\nabla^{\boldsymbol{\alpha}} u(\boldsymbol{y}))\right) = 0$. If $\boldsymbol{H} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is an infinitely smooth diffeomorphism, the transformed function $v(\boldsymbol{x}) = u(\boldsymbol{H}(\boldsymbol{x}))$ satisfies $F\left(\boldsymbol{x}, (\nabla^{\boldsymbol{\alpha}} v(\boldsymbol{x}))\right) = 0$. Show that the relation between F and G is of the form
$$F(\boldsymbol{x}, \boldsymbol{p}) = G(\boldsymbol{H}(\boldsymbol{x}), L(\boldsymbol{x})\boldsymbol{p}), \qquad (L(\boldsymbol{x})\boldsymbol{p})_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta} \in I} L_{\boldsymbol{\alpha}\boldsymbol{\beta}} p_{\boldsymbol{\beta}}$$
where $L(x)$ is an invertible matrix whose coefficients $L_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\boldsymbol{x})$ consist of products of derivatives of $\boldsymbol{H}$ with $L_{\boldsymbol{\alpha}\boldsymbol{\beta}} = 0$ if $|\boldsymbol{\alpha}| < |\boldsymbol{\beta}|$. Show alos that for $\boldsymbol{\alpha}$ with $\alpha_j = m\delta_{ij}$ and $m \leq N$, we have
$$L_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \left(\frac{\partial \boldsymbol{H}}{\partial x_i}\right)^{\boldsymbol{\beta}}, \qquad \forall |\boldsymbol{\beta}| = m$$

Exercise 6:

For some $\boldsymbol{n} \in \mathbb{R}^2$ with $|\boldsymbol{n}| = 1$ let $\Gamma = \{\boldsymbol{y} \in \mathbb{R}^2 : \boldsymbol{y} \cdot \boldsymbol{n} = 0\}$. Consider the initial value problem
$$\sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 u}{\partial y_i \partial y_j} = 0, \qquad u(\boldsymbol{y}) = y_1, \; \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{y}) = y_2, \quad \boldsymbol{y} \in \Gamma$$
Go through Section 1.5. for this example. What are $G, \psi_k, \Phi, \psi, \boldsymbol{H}, F, \varphi_k$ and when is Γ characteristic ...