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SS 2000

Tutorial *Analytical Methods for PDEs*

Return: Monday, 15.05. 1pm, metal box (Kehrwald), ground floor bldg. 48

Exercise 10:

For a constant force field $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{g} \in \mathbb{R}^3$, the transport equation derived in Exercise 9 has the form

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + \boldsymbol{g} \cdot \nabla_{\boldsymbol{v}} f = 0. \tag{1}$$

Consider the initial value problem for (1) with $f(0, \boldsymbol{x}, \boldsymbol{v}) = f^0(\boldsymbol{x}, \boldsymbol{v})$ and solve it with the method of characteristics. (Hint: first write the problem in standard form $\boldsymbol{a}(\boldsymbol{y}) \cdot \nabla u(\boldsymbol{y}) = b(\boldsymbol{y}) u(\boldsymbol{y})$, $\boldsymbol{y} \in \Omega$ and $u(\boldsymbol{y}) = \phi(\boldsymbol{y})$ for $\boldsymbol{y} \in \Gamma$. What is $\boldsymbol{y}, \Omega, \Gamma, u, \phi, \boldsymbol{a}, b$?)

Exercise 11:

Consider the problems on $\mathbb{R}^2$

$$\frac{\partial}{\partial x_1}(x_1 u(\boldsymbol{x})) + \frac{\partial}{\partial x_2}(x_2 u(\boldsymbol{x})) = 0, \tag{2}$$

$$\frac{\partial}{\partial x_1}(x_1 u(\boldsymbol{x})) - \frac{\partial}{\partial x_2}(x_2 u(\boldsymbol{x})) = 0 \tag{3}$$

with values $\hat{u}(\phi)$ prescribed on the unit circle $\Gamma = \{(\cos\phi, \sin\phi) : \phi \in [0, 2\pi)\}$. Sketch the characteristic ground curves and find points on Γ where the normal has a characteristic direction. Determine a local (around Γ) solution of (2) with the method of characteristics. By considering the characteristic ground curves, find maximal subsets of $\mathbb{R}^2$ to which local solutions of (2) and (3) can be extended.

Exercise 12:

Let $\emptyset \neq \Omega \subset \mathbb{R}^d$ and $\emptyset \neq W \subset \mathbb{R}$ be open, $\boldsymbol{a} \in C^1(\Omega \times W, \mathbb{R}^d)$, $b \in C^1(\Omega \times W, \mathbb{R})$. Furthermore, let $\Gamma \subset \Omega$ be a surface, $\phi \in C^0(\Gamma, \mathbb{R})$ and consider the problem

$$\boldsymbol{a}(\boldsymbol{y}) \cdot \nabla u(\boldsymbol{y}) = b(\boldsymbol{y}) u(\boldsymbol{y}), \quad \boldsymbol{y} \in \Omega, \qquad u(\boldsymbol{y}) = \phi(\boldsymbol{y}), \quad \boldsymbol{y} \in \Gamma. \tag{4}$$

Show that (4) admits at most one solution under the assumption that the characteristic ground curves obtained from the characteristic system of (4) with initial values $(\boldsymbol{y}, \phi(\boldsymbol{y}))$, $\boldsymbol{y} \in \Gamma$, reach every point in Ω.

please turn over

Exercise 13:

Consider a simple traffic model with an x-dependent velocity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(V_\epsilon \rho) = 0, \qquad \rho(0,x) = \rho^0(x) \tag{5}$$

where $\rho^0(x) = \exp(-x^2)$ and $V_\epsilon(x) = V_1(x/\epsilon)$ for $\epsilon > 0$ with

$$V_1(x) = \begin{cases} \dfrac{2-3x}{1-x} & x < 0 \\[2ex] \dfrac{x+2}{x+1} & x \geq 0 \end{cases}$$

- Calculate the characteristic ground curves. (Hint: use the parameterization $(t(x), x)$ and not $(t, x(t))$.)
- Sketch the ground curves for very small $\epsilon > 0$ in the (t, x) plane.(Hint: essentially draw the limit case $\epsilon \to 0$.)
- Sketch the solution for very small $\epsilon > 0$ and various $t > 0$. To get an idea of what happens, try a sequence of times $t = \frac{1}{30}, \frac{2}{30}, \frac{3}{30}, \frac{4}{30}, \ldots$ (Hint: Don't calculate anything explicitly here! Only use the conservation property. Think of $\epsilon = 10^{-10000}$ so that, practically, $V_\epsilon' = 0$ apart from a very small interval around the origin $x = 0$.)
- How does the solution of

$$\frac{\partial \rho}{\partial t} + V_\epsilon' \frac{\partial \rho}{\partial x} = 0, \qquad \rho(0,x) = \rho^0(x)$$

differ from the one of (5)?