Dr. Michael Junk
Dipl. Math. techn. Dirk Kehrwald SS 2000

Tutorial *Analytical Methods for PDEs*

Return: Monday, 22.05. 1pm, metal box (Kehrwald), ground floor bldg. 48

Exercise 14:

Use the method of characteristics to construct the solution u of the population model

$$\frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -\mu(a)u(t,a), \qquad t, a > 0$$
$$u(0,a) = u^0(a), \quad a > 0, \qquad u(t,0) = B(t), \quad t \geq 0$$

where $B : [0,\infty) \mapsto \mathbb{R}$ is the birth rate which we first assume to be prescribed. (Sketch also the (t,a)-diagram). The actual birth rate B is connected to the age dependent fertility factor $\lambda(a)$ and the age distribution u according to $B(t) = \int_0^\infty \lambda(a)u(t,a)\,da$. Reformulate this relation as a Volterra integral equation for B, i.e. find K and f in

$$B(t) + \int_0^t K(t-a)B(a)\,da = f(t) \qquad t \geq 0.$$

Exercise 15:

Consider Burgers' equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0, \qquad u^0(x) = \begin{cases} 1 & x \leq -1 \\ -x & -1 < x < 0 \\ 0 & x \geq 0 \end{cases}$$

- Construct a solution $u : (0,1) \times \mathbb{R} \mapsto \mathbb{R}$ with the method of characteristics. Sketch the (t,x)-diagram with the characteristic ground curves.
- Check that u is a classical solution in the sub domains where it is differentiable.
- Show that $u \in C([0,1) \times \mathbb{R})$ is a weak solution in $[0,1) \times \mathbb{R}$ (i.e. by taking test functions in $C_0^1([0,1) \times \mathbb{R})$).
- Continue u to a weak solution on $[0,\infty) \times \mathbb{R}$ by introducing a shock using the Rankine-Hugoniot condition. Sketch the final (t,x)-diagram with shock curve and characteristic ground curves.
- Compare the curve $x \mapsto (x, u(2,x))$ with the "multi-valued solution" $\bar{x} \mapsto (\bar{x} + 2u^0(\bar{x}), u^0(\bar{x}))$ which follows from the method of characteristics.

Exercise 16:

Consider again Burgers' equation but now with the initial value

$$u^0(x) = \begin{cases} 0 & x < -2 \\ x+2 & -2 \leq x \leq -1 \\ -x & -1 < x < 0 \\ 0 & x \geq 0 \end{cases}$$

Find a weak solution using the method of characteristics and the Rankine-Hugoniot condition. (Note: the shock curve is not a straight line in this case.) Plot the shock speed over time and sketch the solution for various t.