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Tutorial *Analytical Methods for PDEs*

Return: Monday, 29.05. 1pm, metal box (Kehrwald), ground floor bldg. 48

Exercise 17:

In the non-linear traffic model

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} Q(\rho) = 0, \qquad \rho(0,x) = \rho^0(x)$$

we assume $Q(\rho) = V_{car}(\rho)\rho$ with $V_{car}(\rho) = 1 - \rho/\rho_{max}$ for $\rho \in [0, \rho_{max}]$ (the maximal car velocity has been set to one by choosing suitable length or time units) and

$$\rho^0(x) = \begin{cases} \rho_{max}(x+1) & -1 < x < 0 \\ \rho_{max}(1-x) & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

- Plot the functions $V_{car}(\rho), Q(\rho), Q'(\rho)$ and $\rho^0(x), Q'(\rho^0(x))$.
- Use the method of characteristics to calculate the solution up to the shock time. Sketch the (t,x)-diagram.
- Continue the solution by introducing a shock curve (using the ODE following from the Rankine-Hugoniot condition).
- Sketch the solution for several t before and after the shock time. Interpret the traffic situations.

Exercise 18:

Find a weak solution of Burgers' equation with initial value

$$u^0(x) = \begin{cases} 4 & x \leq -1 \\ 2 & -1 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

Plot characteristic ground curves and shock curves in the (t,x)-diagram. Sketch the solution at times $t = 1/2, 1, 3/2$.

Exercise 19:

Consider again Burgers' equation but now with the initial value

$$u^0(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and show that

$$u(t,x) = \begin{cases} 0 & x < \frac{1}{2}t \\ 1 & x \geq \frac{1}{2}t \end{cases}$$

is a weak solution which does not satisfy the entropy condition for $\eta(u) = u^2/2$. In fact, the unique entropy solution is given by a "rarefaction wave" which is constructed by assuming that the solution u depends only on x/t. Derive an equation for $\hat{u} : \mathbb{R} \mapsto \mathbb{R}$ by plugging $u(t,x) = \hat{u}(x/t)$ into Burgers' equation. Show that the resulting function u is a weak solution.