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SS 2000

Tutorial *Analytical Methods for PDEs*

Return: Monday, 05.06. 1pm, metal box (Kehrwald), ground floor bldg. 48

Exercise 20:

Find $\boldsymbol{U} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ which satisfies the linear problem

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial x_1} + \begin{pmatrix} 5 & 3 \\ 2 & 2 \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial x_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{U}(0, x_2) = \boldsymbol{U}^0(x_2).$$

Hint: first convert the equation in a form used in the lecture with x_1 as "t-variable".

Exercise 21:

Rewrite the initial value problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = u^0(x), \quad \frac{\partial u}{\partial t}(0,x) = v^0(x)$$

as linear first order problem in the variables $U_1 = u$, $U_2 = \frac{\partial u}{\partial x}$, $U_3 = \frac{\partial u}{\partial t}$. After decoupling the system in a basis of eigenvectors, use the method of characteristics to find the solution. Show that d'Alembert's formula for u is recovered.

Exercise 22:

Rewrite the system of Maxwell equations

$$\begin{aligned} \frac{\partial \boldsymbol{E}}{\partial t} - \frac{1}{\epsilon\mu} \operatorname{curl} \boldsymbol{B} + \frac{\gamma}{\epsilon} \boldsymbol{E} &= -\frac{1}{\epsilon} \boldsymbol{J} \\ \frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{curl} \boldsymbol{E} &= 0 \end{aligned} \tag{1}$$

as a system in $\boldsymbol{U} = (E_1, E_2, E_3, B_1, B_2, B_3)^T$ and show that it is hyperbolic. Here, $\boldsymbol{E}(t, \boldsymbol{x}) \in \mathbb{R}^3$ is the electric field, $\boldsymbol{B}(t, \boldsymbol{x}) \in \mathbb{R}^3$ is the magnetic induction, $\boldsymbol{J}(t, \boldsymbol{x})$ is the vector of the current density of external sources and the positive, constant scalars ϵ, μ, γ are the permitivity, the permeability and the conductivity of the medium.

The full set of Maxwell equations consists, in addition to (1), of

$$\epsilon \operatorname{div}_{\boldsymbol{x}} \boldsymbol{E} = \rho, \qquad \operatorname{div}_{\boldsymbol{x}} \boldsymbol{B} = 0$$

where the unknown $\rho(t, \boldsymbol{x})$ is the density of free charges. After solving (1), ρ can obviously be calculated directly based on $\boldsymbol{E}$. Using (1), show that the charge satisfies the balance law

$$\frac{\partial \rho}{\partial t} + \operatorname{div}_{\boldsymbol{x}} \boldsymbol{J} = -\frac{\gamma}{\epsilon} \rho.$$

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With the second condition $\operatorname{div}_{\boldsymbol{x}} \boldsymbol{B} = 0$, the Maxwell equations are over-determined if the initial values

$$\boldsymbol{E}(0, \boldsymbol{x}) = \boldsymbol{E}^0(\boldsymbol{x}), \qquad \boldsymbol{B}(0, \boldsymbol{x}) = \boldsymbol{B}^0(\boldsymbol{x})$$

satisfy $\operatorname{div}_{\boldsymbol{x}} \boldsymbol{B}^0 = 0$ (show that under this condition on $\boldsymbol{B}^0$, the relation $\operatorname{div}_{\boldsymbol{x}} \boldsymbol{B} = 0$ holds automatically for the solution $\boldsymbol{B}$ of (1)).

The equation (1) of an electromagnetic field are often transformed to second order equations. Using $\operatorname{curl} \operatorname{curl} \boldsymbol{A} = \nabla \operatorname{div} \boldsymbol{A} - \Delta \boldsymbol{A}$ which relates the double curl of a smooth vector-field $\boldsymbol{A} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ to its Laplacian, show that $\boldsymbol{B}$ satisfies the "telegraph equation"

$$\frac{\partial^2 \boldsymbol{B}}{\partial t^2} - \frac{1}{\epsilon\mu} \Delta \boldsymbol{B} + \frac{\gamma}{\epsilon} \frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{\epsilon} \operatorname{curl} \boldsymbol{J}.$$