Dr. Michael Junk
Dipl. Math. techn. Dirk Kehrwald

SS 2000

Tutorial *Analytical Methods for PDEs*

Return: Monday, 19.06. 1pm, metal box (Kehrwald), ground floor bldg. 48

Exercise 23:

a) Derive the formula for the surface area $\omega_d = |\partial B_1(0)|$ of the unit sphere in $\mathbb{R}^d$

$$\omega_d = \frac{2\Gamma\left(\frac{1}{2}\right)^d}{\Gamma\left(\frac{d}{2}\right)}, \qquad \Gamma(x) = \int_0^\infty t^{x-1}\exp(-t)\,dt, \quad x > 0.$$

(Hint: integrate $\exp(-|x|^2)$ in Cartesian and spherical coordinates.)

b) Show that the volume $|B_r(0)|$ of a ball of radius r in $\mathbb{R}^d$ is $\omega_d r^d/d$.

c) Let u be a continuous function on $B_{\tilde{R}}(0)$. After going over to spherical coordinates, calculate the derivative

$$\frac{d}{dr}\int_{B_r(0)} u(y)\,dy\bigg|_{r=R}, \qquad 0 < R < \tilde{R}.$$

d) Use (b) and (c) to show that the mean value property over balls

$$u(0) = \frac{1}{|B_R(0)|}\int_{B_R(0)} u(y)\,dy, \qquad \forall\, 0 < R < \tilde{R}$$

implies the mean value property over spheres

$$u(0) = \frac{1}{|\partial B_R(0)|}\int_{\partial B_R(0)} u(y)\,d\sigma(y), \qquad \forall\, 0 < R < \tilde{R}.$$

Exercise 24:

Find all radially symmetric solutions $u : \mathbb{R}^3 \mapsto \mathbb{R}$ of the equations

$$\Delta u + u = 0 \qquad \text{and} \qquad \Delta u - u = 0.$$

(Hint: Look for a spherically symmetric functions $u(x) = f(|x|) = \hat{f}(|x|)/|x|$.)

a) Extend the result to the case $\Delta \pm \omega^2$, $\omega > 0$ by a simple transformation of coordinates (Remark: $\Delta + \omega^2$, $\omega \in \mathbb{C}$, is referred to as *Helmholtz operator* but sometimes this name is restricted to the case $\omega \in \mathbb{R}$.)

b) Show that spherically symmetric solutions of the Laplace equation are recovered in the limit case $\omega = 0$.

Exercise 25:

Prove the generalized Euler-Poisson-Darboux equation presented in the lecture. (Hint: All major ingredients of the proof have been discussed already in Exercise 8. The trick is now to use $\int_0^r s^{d-1} I_u(\boldsymbol{x}, s)\,ds$ and the divergence theorem.)