List of Axioms

The concept of real numbers is introduced by specifying a list of properties (axioms).

short version: The set $\mathbb{R}$ of real numbers is a **complete**, **ordered field**.

field axioms

The set $\mathbb{R}$ is a field, i.e. there are mappings

$$\begin{array}{rcl} \mathbb{R}\times\mathbb{R} & \to & \mathbb{R} \\ (x,y) & \to & x+y \end{array} \qquad\qquad \begin{array}{rcl} \mathbb{R}\times\mathbb{R} & \to & \mathbb{R} \\ (x,y) & \to & xy \end{array}$$

called addition and multiplication such that

A1 $x+(y+z)=(x+y)+z$ for all $x,y,z\in\mathbb{R}$

A2 $x+y=y+x$ for all $x,y\in\mathbb{R}$

A3 there exists $0\in\mathbb{R}$ such that $x+0=x$ for all $x\in\mathbb{R}$

A4 for each $x\in\mathbb{R}$ there exists $(-x)\in\mathbb{R}$ such that $x+(-x)=0$

M1 $x(yz)=(xy)z$ for all $x,y,z\in\mathbb{R}$

M2 $xy=yx$ for all $x,y\in\mathbb{R}$

M3 there exists $1\in\mathbb{R}$, $1\neq 0$, such that $1\,x=x$ for all $x\in\mathbb{R}$

M4 for each $x\neq 0$, there exists $x^{-1}\in\mathbb{R}$ such that $xx^{-1}=1$

DL $x(y+z)=xy+xz$ for all $x,y,z\in\mathbb{R}$

Properties **A1** and **M1** are called the *associative laws*, **A2** and **M2** the *commutative laws*, and **DL** the *distributive law.*

order axioms

The field $\mathbb{R}$ is ordered, i.e. there exists a relation "$\leq$" on $\mathbb{R}$ such that for all $x,y,z\in\mathbb{R}$

O1 either $x\leq y$ or $y\leq x$

O2 if $x\leq y$ and $y\leq x$ then $x=y$

O3 if $x\leq y$ and $y\leq z$ then $x\leq z$

O4 if $x\leq y$ then $x+z\leq y+z$

O5 if $x\leq y$ and $0\leq z$ then $xz\leq yz$

Property **O3** is called the *transitive law.* We use $x\geq y$ synonym for $y\leq x$ and, if $x\neq y$, we also write $x<y$ for $x\leq y$, respectively $x>y$ for $x\geq y$.

completeness axiom

The ordered field $\mathbb{R}$ is complete, i.e.

C1 Let S be a non-empty subset of $\mathbb{R}$ and $U=\{x\in\mathbb{R}: s\leq x \text{ for all } s\in S\}$ the associated set of upper bounds. If S is bounded above (i.e. $U\neq\emptyset$) then S has a smallest upper bound (i.e. there exists $x_0\in U$ and $x_0\leq x$ for all $x\in U$).