Miscellaneous Exercises *Analysis I*

Return: Monday 7.01.02

Exercise 1:

Let (a_n) and (b_n) be Cauchy sequences. Prove or disprove

(i) $a_n + b_n$ is Cauchy

(ii) $a_n - b_n$ is Cauchy

Exercise 2:

Show that if (a_n) converges and (b_n) diverges, then $a_n + b_n$ diverges.

Exercise 3:

If (a_n) (b_n) and (c_n) are three sequences such that

(i) $a_n \leq b_n \leq c_n$ for all n

(ii) $\lim a_n = \lim c_n = \text{l}$.

Show that $\lim b_n = \text{l}$.

Exercise 4:

Show that the sequence defined by

$$f(1) = \sqrt{2}$$

$$f(n+1) = \sqrt{2f(n)}$$

converges. (Hint: Use monotone convergence theorem.)

Exercise 5:

If (a_n) is a sequence such that

$$\lim \frac{a_{n+1}}{a_n} = l, \text{ where } |l| < 1$$

Then show that $\lim a_n = 0$.

Exercise 6:

Show that $\lim n^{1/n} = 1$. (Hint: Observe $n^{1/n} = 1 + h_n$ and use binomial theorem to prove that $\lim h_n = 0$.)

Exercise 7:

Let (a_n) and (b_n) be convergent sequences such that

$$a_n < b_n \text{ for all } n.$$

Show that $\lim a_n \leq \lim b_n$.

Exercise 8:

Prove that the sequence defined by

$$f(1) = \sqrt{7} \qquad f(n+1) = \sqrt{7 + f(n)}$$

converges to the positive square root of the equation $x^2 - x - 7 = 0$.

Exercise 9:

In what follows justify your answers

(i) Give an example of two series $\sum a_n$ and $\sum b_n$ such that both $\sum a_n$ and $\sum b_n$ diverge but their difference converges.

(ii) Check for the convergence of $\sum \cos(n)$.

(iii) Give an example of a series which is convergent but not absolutely convergent.

(iv) Can you construct an absolutely convergent series which is not convergent.

Exercise 10:

Can you construct a sequence of continuous functions which converges to a discontinuous function?

Wish you all a
Merry Christmas
and a
Wonderful New Year !