Dr. M. Junk WS 2001/02-11

Tutorial *Analysis I*

Return: 12:00, Friday 01.02.2002

Exercisee 43:

Use the definition $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ to show that $\exp(x) > 1$ for $x > 0$. Then, prove with the relation $\exp(x+y) = \exp(x)\exp(y)$ that

(i) $\exp(x) < 1$ for $x < 0$,

(ii) exp is strictly increasing,

(iii) $\inf_{x\in\mathbb{R}} \exp(x) = 0$, $\sup_{x\in\mathbb{R}} \exp(x) = \infty$.

Justify that exp is invertible with a continuous inverse (usual name: ln). Show that the relation $\ln(xy) = \ln x \ln y$ holds for $x, y > 0$.

Exercisee 44:

Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Moreover, assume that $f(1) > 0$ and that f is continuous in at least one point.

(i) Show: f is continuous on $\mathbb{R}$, $f(0) = 1$ and $f(x) > 0$ for all $x \in \mathbb{R}$.

(ii) Express in terms of $f(1)$: first $f(n)$ for $n \in \mathbb{N}$, then $f(p)$ for all $p \in \mathbb{Z}$, then $f(1/q)$ for $q \in \mathbb{N}$, and finally $f(p/q)$ for $p \in \mathbb{Z}, q \in \mathbb{N}$.

(iii) Use continuity of f to show that $f(x) = \exp(x \ln f(1))$.

By the way: for $a > 0$, we define $a^x = \exp(x \ln a)$, $x \in \mathbb{R}$.

Exercisee 45:

Let (g_n) be a uniform convergent sequence of functions $g_n : S \to \mathbb{R}$ with limit $g : S \to \mathbb{R}$, where $\emptyset \neq S \subset \mathbb{R}$. Let (x_n) be a sequence in S which converges to $x \in S$. Show that $g_n(x_n) \to g(x)$.

Exercisee 46:

Let (M, d) be a complete metric space consisting of a vector space M and a translation invariant metric d (i.e. $d(x,y) = d(x-z, y-z)$ for all $x, y, z \in M$). For every $n \in \mathbb{N}_0$ let there be sequences $(g_{nk})_{k\in\mathbb{N}}$ which converge in M for $k \to \infty$. Moreover, assume that $d(g_{nk}, 0) \le b_n$ for all k, n and $\sum b_n < \infty$. Show:

(i) the series $\sum_{n=0}^{\infty} g_{nk}$ converge for every $k \in \mathbb{N}$

(ii) $\lim_{k\to\infty} \sum_{n=0}^{\infty} g_{nk}$ exists and

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} g_{nk} = \sum_{n=0}^{\infty} \lim_{k\to\infty} g_{nk}.$$

Hint: Take the case $M = \mathbb{R}$ and $d_{\mathbb{R}}(x,y) = |x-y|$ as guiding line. Use translation invariance to show that d behaves essentially like $d_{\mathbb{R}}$.