Tutorial *Analysis I*

Return: 12:00, Friday 08.02.2002

Exercise 47:

Show that for $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ and an arbitrary polynomial P

$$\lim_{x\to\infty} P(x)/\exp(x) = 0.$$

Use the definition of the derivative to show the relation $\exp'(0) = 1$ and conclude that $\exp' = \exp$ by using the functional equation $\exp(x+y) = \exp(x)\exp(y)$. Let now $f(x) = \exp(-1/x)$ for $x > 0$. Calculate the first three derivatives of f. Show that f is infinitely often differentiable and that each derivative has the structure $P(1/x)f(x)$ with P being a polynomial. Conclude that the functions

$$\bar{f}(x) = \begin{cases} \exp\left(-\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}, \qquad \hat{f}(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

are infinitely often differentiable. Sketch the functions $\bar{f}$ and $\hat{f}$.

Exercise 48:

The functions sin and cos are given by power series (see Exercise 41).

(i) Differentiate the power series term by term to prove that $\sin' = \cos$ and $\cos' = -\sin$.

(ii) Conclude from (i) that the function $x \mapsto \sin^2 x + \cos^2 x$ is constant on $\mathbb{R}$ and then, determine the constant value.

(iii) Let $\pi/2 > 0$ be the smallest positive root of cos. Show that the function

$$\tan x = \frac{\sin x}{\cos x}, \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

is strictly increasing and determine the derivative of the inverse function arctan. Specify also the domain of definition of arctan.

Exercise 49:

Let f be differentiable on the open interval (a,b) and assume that the derivative f' is bounded. Show that f is uniformly continuous on (a,b).

Exercise 50:

Let $f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show: f is differentiable and $f'(0) > 0$, but in every interval $(-\epsilon, \epsilon)$, $\epsilon > 0$, f is *not* strictly increasing. Moreover, f' is not continuous in $x = 0$.