Dr. M. Junk WS 2001/02-2

Tutorial *Analysis I*

Return: 12:00, Friday 16.11.

Exercise 6:

Let $A \neq \emptyset$ be a subset of $\mathbb{R}$ with a finite number of elements. Show that A has a maximum and a minimum.

Exercise 7:

Let $n!$ (n factorial) denote the product of the first n natural numbers, i.e. $n! = 1 \cdot 2 \cdot 3 \cdots n$, $n \in \mathbb{N}$, and set $0! = 1$. The *binomial coefficients* are defined for $k, n \in \mathbb{Z}$

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise} \end{cases}.$$

Show that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \tag{1}$$

and prove with (1) that for $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$ the *binomial theorem* holds, i.e.

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n. \tag{2}$$

Note that, using the sum notation $\alpha_0 + \alpha_1 + \cdots + \alpha_n = \sum_{k=0}^{n} \alpha_k$, we can also write (2) in the form

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Exercise 8:

Let $A(n,k)$ be the number of subsets having k elements of a set with n elements where $k, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Show that

$$A(n,k) + A(n,k+1) = A(n+1,k+1).$$

Does this imply that $A(n,k) = \binom{n}{k}$? If yes, why? If no, why not?

Exercise 9:

Construct a set of propositions P_n, $n \in \mathbb{N}$ for which one can show the induction step (i.e. $P_n \Rightarrow P_{n+1}$), although not all propositions are correct (i.e. the induction basis is an essential part of induction proofs). Try to find "interesting" propositions P_n.

Exercise 10:

Show that:

(i) If $A \neq \emptyset$ is a subset of $\mathbb{R}$ which is bounded above, then for every $\epsilon > 0$ there exists $a_\epsilon \in A$ such that $a_\epsilon > \sup A - \epsilon$.

(ii) If $A \neq \emptyset$ is a subset of $\mathbb{R}$ which is bounded below, then for every $\epsilon > 0$ there exists $a_\epsilon \in A$ such that $a_\epsilon < \inf A + \epsilon$.

(iii) If $a, b \in \mathbb{R}$ are numbers with the property $a - \epsilon \leq b \leq a + \epsilon$ for every $\epsilon > 0$, then $a = b$.

Use these results to show for bounded sets A, B and their sum and product

$$A + B := \{a + b : a \in A,\ b \in B\}, \qquad A \cdot B := \{ab : a \in A,\ b \in B\}$$

the following statements:

(i) $\sup(A + B) = \sup A + \sup B$, $\inf(A + B) = \inf A + \inf B$

(ii) $\sup(A \cdot B) = \sup A \sup B$, $\inf(A \cdot B) = \inf A \inf B$, if all elements of A, B are non-negative.