Tutorial *Analysis I*

Return: 12:00, Friday 30.11.

Exercise 15:

Let $a \geq 0$ and $m \in \mathbb{N}$. Show that the equation $x^m = a$ has at most one non-negative solution $x \in \mathbb{R}$.
To construct a sequence of approximate solutions $x_0, x_1, \ldots$ of $x^m = a$, the following algorithm is proposed (the construction of the first four terms is demonstrated for the problem $x^2 = 5$):

$0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9 \to x_0 = 2$.

$2.1^2 = 4.41$, $2.2^2 = 4.84$, $2.3^2 = 5.29 \to x_1 = 2.2$.

$2.21^2 = 4.8841$, $2.22^2 = 4.9284$, $2.23^2 = 5.49729$, $2.24^2 = 5.0176 \to x_2 = 2.23$.

$2.231^2 = 4977361$, $\ldots$, $2.236^2 = 4.999696$, $2.237^2 = 5.004169 \to x_3 = 2.236$.

Define the sequence $(x_n)_{n \in \mathbb{N}_0}$ for the case $x^m = a$ in a mathematically precise way and prove that your construction is possible for every $n \in \mathbb{N}$. Then, show that the sequence converges to a solution of the equation $x^m = a$.

Exercise 16:

Show that: a sequence (s_n) converges to $s \in \mathbb{R}$ if and only if every subsequence of (s_n) contains a subsequence which converges to s.

Exercise 17:

Let $N \in \mathbb{N}$, let (s_n), (t_n) be arbitrary sequences in $\mathbb{R}$ and (r_n), (p_n) be bounded sequences. Show that:

(i) $\liminf s_n = -\limsup(-s_n)$

(ii) $\liminf s_n \leq \liminf t_n$, $\limsup s_n \leq \limsup t_n$, if $s_n \leq t_n$ for all $n > N$

(iii) $\liminf(s_n + r_n) \geq \liminf s_n + \liminf r_n$, $\limsup(s_n + r_n) \leq \limsup s_n + \limsup r_n$

(iv) $\limsup(p_n r_n) \leq \limsup p_n \limsup r_n$, if $p_n, r_n \geq 0$

Give examples of sequences for which strict inequality holds in (iii) and (iv).